From sectional to Ricci curvature via symmetry

Lee Kennard (Syracuse University)
Lawrence Mouillé (Trinity University)
Jan Nienhaus (UCLA)

Workshop on Curvature and Global Shape

July 28 – August 1, 2025

Intermediate Ricci curvature

sec $=$ Ric_1

Ric $=$ Ric_{n-1}

Definition: For $x \in T_pM$ and a k-plane $\pi \subseteq (\mathbb{R}x)^\perp$, define

$$\text{Ric}_k(x,\pi) = \sum_{i=1}^{k} \sec(x, y_i)$$

where $\{y_1, \ldots, y_k\}$ is any o.n.b. for π.

[Bishop-Crittenden '64], [Hartman '79], $\sim$ 5 papers in the '80s, $\sim$ 1 paper/year in the '90s, '00s, and '10s (many not really about Ric_k), and $>$ 30 so far in the '20s.

Constant, non-negative, and positive curvature

sec $= \kappa$ implies M is a space form.

$\mathrm{Ric}_k =$ constant for any $1 \le k < n-1$ implies sec $= \kappa$.

$\mathrm{Ric} = (n-1)\kappa$ . . . many examples & no classification.

sec ≥ 0 implies $\pi_1(M)$ is f.g., $\dim H^*(M^n; \mathbb{Z}_p) \le C(n)$, and $\hat{A} = 0$.

(Reiser-Wraith '23) Gromov's Betti bound fails for $\frac{n}{2} \lesssim k \le n-1$.

Ric ≥ 0 still yields structural results but the above statements have counterexamples: [Bruè-Naber-Semola '25], [Sha-Yang '89].

dim > 24: only known s.c. closed sec > 0 examples are $\mathbb{S}, \mathbb{CP}, \mathbb{HP}$.

Surprising fact: For any $k \ge 1$ and dim $> n(k)$, the only known simply connected, closed examples with $\mathrm{Ric}_k > 0$ are $\mathbb{S}$, $\mathbb{CP}$, and $\mathbb{HP}$.

For dim $\lesssim 2k$, there are lots, e.g., Ric > 0 and sym. spaces [Amann-Quast-Zarei, Domínguez-Vázquez-González-Álvaro-Mouillé, R-W].

Positive Ric_2 with symmetry

Dimension four:

$\mathbb{S}^2 \times \mathbb{S}^2$ admits $\mathrm{Ric}_2 > 0 + \mathrm{SO}(3)$ symmetry (Müter, Wilking)

(Neither Hopf's conj. nor Hsiang-Kleiner's theorem extends to Ric_2.)

$\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ do not admit $\mathrm{Ric}_2 > 0 + T^2$ sym. [KM '24]

Q: Does $\mathbb{CP}^2 \# \mathbb{CP}^2$ admit $\mathrm{Ric}_2 > 0$ (with T^2 or sym.)?

Q: Do $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ admit $\mathrm{Ric}_2 > 0$ (with cohom 1 symmetry)?

Dimension five:

$\mathbb{S}^2 \times \mathbb{S}^3$ admits $\mathrm{Ric}_2 > 0 + \mathrm{T}^2$ sym. (Wilking)

(Rong's classification of T^2-actions on M^5 does not extend to Ric_2.)

Q: Does the non-trivial $\mathbb{S}^3$-bundle over $\mathbb{S}^2$ admit $\mathrm{Ric}_2 > 0$?

Q: Does the Wu manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$ admit $\mathrm{Ric}_2 > 0$?

Q: Does $\mathrm{Ric}_2 > 0$ and T^2 symmetry imply $b_2(M) \leq 1$?

Positive Ric_2 with symmetry, II

Maximal symmetry rank:
(Grove-Searle '94) M^n admits sec > 0 + T^d symmetry $\Rightarrow d \leq \lfloor \frac{n+1}{2} \rfloor$.
Moreover, equality only holds if M is diffeomorphic to S^n and $\mathbb{CP}^{n/2}$.

(M '22, KM '24) $\mathrm{Ric}_2 > 0 + \mathrm{T}^d \Rightarrow$ same bound and homeo. rigidity.

Q: Equivariant diffeomorphism rigidity?

Q: Dimension 4?

Q: In dim 6, the classification needs $\chi(M) \neq 0$. Can we prove it?

(M '22, $k \geq 3$) $\mathrm{Ric}_k > 0$ and T^d symmetry $\Rightarrow d \leq \left\lfloor \frac{n+k-2}{2} \right\rfloor$.

Q: Optimality holds for $k \leq 3$. How about for $k = 4, 5, \ldots$?

Q: It also holds for $k = n-1$ (Reiser). How about for $k \leq n-2$?

Hopf's Euler characteristic positivity conjecture

Conjecture (Hopf 1930s): sec > 0 implies $\chi(M^{2n}) > 0$?

(K-Wiemeler-Wilking, N): sec > 0 + T^4 symmetry $\Rightarrow \chi(M^{2n}) > 0$.

(KWW '25) sec > 0 + T^9 symmetry + c.i.g. $\Rightarrow M^{2n} \sim_{\mathbb{Q}} \mathbb{S}, \mathbb{CP}, \mathbb{HP}$.

How about $\mathrm{Ric}_2 > 0$? We have **homogeneous examples**:

$\mathbb{S}^3 \times \mathbb{S}^3$	(Wilking)
$\mathbb{S}^7 \times \mathbb{S}^7$	(DeVito, Domínguez-Vázquez, González-Álvaro, Rodríguez-Vázquez)

Cor: $\mathrm{Ric}_2 > 0$ does not imply $\chi(M^{2n}) > 0$ (even with T^4 symmetry)

Thm (KMN): If M^{4n} has $\mathrm{Ric}_2 > 0$ + T^{10} symmetry, then $\chi(M) > 0$.

Thm (KMN): If also the action has c.i.g., then $H^{\mathrm{odd}}(M; \mathbb{Q}) = 0$.

Proof sketch for Theorem 1 (geometric part)

Setup: M^{4n} has $\mathrm{Ric}_2 > 0$ and isometric action by T^{10}.

(Mouillé) There exists T^8 with non-empty fixed-point set.

We want to show $\chi(F) > 0$ for all components $F \subseteq M^{\mathrm{T}^8}$.

(Borel formula) There is T^7 with $F \subseteq G = M_p^{\mathrm{T}^7}$ with $\dim G \equiv 0\ (4)$.

(S^1 splitting) There is $G \subseteq P \subseteq Q^m$ such that $\dim P \equiv 0\ (4)$ and the inclusion $P \subseteq Q$ is $(\dim P - 1)$-connected with codim $\leq \frac{1}{3}(m-1)$.

(Partial Four Periodicity Theorem) P has four-periodic Betti numbers on degrees $1 \leq * \leq n-1$, so $b_1(P) \geq b_5(P)$ and $b_i(P) = b_{i+4}(P)$.

By Myers' theorem, $b_1(P) = 0$.
By four-periodicity, all $b_{4i+1}(P) = 0$.
By Poincaré duality, all $b_{4i+3}(P) = 0$.
By Connor-Floyd, F also has vanishing odd Betti numbers.

Proof sketch (algebraic part)

Periodicity Lem. (Wilking '03) If $P^{n-k} \subseteq Q^n$ is a $(\dim P)$-connected inclusion of PD manifolds with $2k \leq n$, then $H^*(Q; \mathbb{Z})$ is k-periodic.

Four Periodicity Thm. (K '13, Nienhaus Ph.D.): If $3k \leq n$ or the normal bundle of P is complex, then $H^*(Q; \mathbb{Q})$ is 4-periodic.

Proof sketch (algebraic part), II

Corollary: If $P^{n-k} \subseteq Q^n$ is a $(\dim P)$-connected inclusion of closed orientable manifolds with $3k \leq n$, then $H^*(Q; \mathbb{Q})$ is 4-periodic.

(Connectedness Lemma + Partial Periodicity)
$\sec > 0$ + torus symmetry $\Rightarrow$ periodicity in degrees $0 \leq * \leq n$.
$\mathrm{Ric}_2 > 0$ + torus symmetry does also but only on $1 \leq * \leq n-1$.

What periodicity results carry over to partial periodicity?

(Nienhaus M.Sc.)

1) Everything in the special case of *irreducible* periodicity.
2) Periodic rings *decompose* into irreducibly periodic rings.

(KMN) If $P^{n-k} \subseteq Q^n$ is a $(\dim P - 1)$-connected inclusion of closed s.c. manifolds with $3k \leq n-1$, then $H^{1 \leq * \leq n-1}(Q; \mathbb{Q})$ is 4-periodic.

What about $\dim M \equiv 2\ (4)$?

Examples: We need to be careful: $S^p \times S^p$ admits $\mathrm{Ric}_2 > 0$ and torus symmetry for $2p \in \{6, 14\}$, so odd Betti numbers need not vanish.

Q: Does there exist M^{4n} with $\mathrm{Ric}_2 > 0$ and $H^{odd}(M; \mathbb{Q}) \neq 0$?

Proof analysis: We can show $0 = b_1 = b_5 = \ldots$ and $b_3 = b_7 = \ldots$ for P, but we need something like the following for $\mathrm{Ric}_2 > 0$:

(b_3 Lemma, [KWW]) If P^{4n+2} is a closed orientable manifold with 4-periodic $\mathbb{Q}$-cohomology, and if there is an S^1-action s.t. $b_1(F_i) = 0$ for all f.p.c. and some $F_0 \subseteq P$ is 7-connected, then $b_3(M) = 0$.

Q: The proof uses Bredon cohomology, but for $\mathrm{Ric}_2 > 0$ we have less control over $H^{even}(M; \mathbb{Q})$. Can we still extend this lemma?

($\bar{b}_3$ Lemma, [K '13]) If P^{4n+2} is a closed, orientable manifold with 4-periodic $\mathbb{Z}_2$-cohomology, then $b_3(M; \mathbb{Z}_2) = 0$.

Q: Is there a Four Periodicity Theorem with $\mathbb{Z}_2$ coefficients?

What about $\text{Ric}_k > 0$ for $k \geq 3$?

Examples: The only simply connected, closed manifolds M^n known to admit $\text{Ric}_k > 0$ with $k << n$ are the rank one symmetric spaces.

Q: Does $\text{Ric}_k > 0$ and $\text{T}^{d(k)}$ symmetry imply $\chi(M^{4n}) > 0$?

Proof analysis $\Rightarrow$ this needs new ideas:

The proofs for $k \leq 2$ relied on the Connectedness Lemma being strong enough to relate b_1 ($= 0$ by Myers) to higher odd Bettis.

The best we expect from a fully general Partial Four Periodicity Theorem is 4-periodic Betti numbers on $k - 1 \leq * \leq n - (k - 1)$.

Q: For $k \geq 3$, can we prove $b_i(M) = 0$ for odd degrees $i \leq k$?