Torus representations with connected isotropy groups: Structure results and applications

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First, the applications . . .

Positive curvature (Toponogov): A Riemannian manifold M has sec > 0 if every geodesic triangle has angle sum $\alpha + \beta + \gamma > \pi$.

Examples: In dim > 24, we only know $\mathbb{S}^n$, $\mathbb{CP}^n$, and $\mathbb{HP}^n$.

Grove Symmetry Program (1990s): Study sec > 0 with symmetry.

(Homogeneous spaces, cohomogeneity one manifolds, quotients, . . .)

Constructions:

- $\pi_1(M) \neq$ spherical space form groups [Sha98, Baz99, GS00, GSZ06]
- a manifold with sec > 0 [Dea11, GVZ11]
- manifolds with sec ≥ 0, including all 5 homotopy $(\mathbb{S}^2 \times \mathbb{S}^2)/\mathbb{Z}_2$ [Tor19] and all 28 homotopy $\mathbb{S}^7$ [GM74, GZ00, GKS20]
- . . . almost/quasi-positive curvature [PW99, Wil01, Wil02, Tap03, EK08, Ker11, Ker12, KT14, DRRW14, DeV18, DN20, DeV21]
- . . . positive bi-orthogonal curvature [Bet17, ST20]

Obstructions: We focus today on torus symmetry: $\mathsf{T}^d \subseteq \mathrm{Isom}(M)$.

Topological rigidity: Positive curvature & torus symmetry

Setup:

(M^n, g) – closed, $\pi_1(M) = 1$, $\sec > 0$ (e.g., $\mathbb{S}^n$, $\mathbb{CP}^{\frac{n}{2}}$, $\mathbb{HP}^{\frac{n}{4}}$)

If M^n admits T^d symmetry, what can we conclude about its topology?

- (Grove-Searle '94) Diffeomorphism rigidity if $d \geq \frac{n}{2}$.
- (Fang-Rong '05) Homeomorphism rigidity if $d \geq \frac{n}{2} - 1$ $(n \geq 8)$.
- (Wilking '03, [DW04]) Homotopy rigidity if $d \geq \frac{n}{4} + 1$ $(n \geq 10)$.
- (Wilking '03) $\mathbb{Q}$-cohomology rigidity* if $d \geq \frac{n}{6} + 1$ $(n \geq 6000)$.
- (K.-Wiemeler-Wilking) $\mathbb{Q}$-cohomology rigidity if $d \geq 9$ and c.i.g.
- (K.-Wiemeler-Wilking [KWW], Nienhaus) Assume n is even.
  - $\mathbb{Q}$-cohomology rigidity if $d \geq 6$ and $H^{\mathrm{odd}}(M; \mathbb{Q}) = 0$.
  - Euler characteristic positivity if $d \geq 4$.

Proof sketch: Reduction to the Main Lemma

Study the isotropy representation $\mathsf{T}^9 \to \mathrm{SO}(T_x M)$ at a fixed point x.

By assumption, this representation has connected isotropy groups.

Main Lemma: There exist $\mathsf{S}^1 \subseteq \mathsf{T}^2 \subseteq \mathsf{T}^3 \subseteq \mathsf{T}^4 \subseteq \mathsf{T}^9$ such that

$$\underbrace{M_x^{\mathsf{T}^4} \hookrightarrow M_x^{\mathsf{T}^3} \hookrightarrow M_x^{\mathsf{T}^2} \hookrightarrow M_x^{\mathsf{S}^1}}_{\dim M_x^{\mathsf{T}^i} \geq \frac{2}{3} \dim M_x^{\mathsf{T}^{i-1}}} \underbrace{\hookrightarrow M}_{\dim M_x^{\mathsf{S}^1} \geq \frac{3}{4} n}$$

Significance of $\frac{2}{3}$ and $\frac{3}{4}$? By the Connectedness Lemma [Wil03]...
$\Longrightarrow$ If $M_x^{\mathsf{T}^4}$ is a $\mathbb{Q}$-cohomology $\mathbb{S}$, $\mathbb{CP}$, or $\mathbb{HP}$, then so is M.

T^4-**theorem** ([KWW], Nienhaus): $M_x^{\mathsf{T}^4}$ is a $\mathbb{Q}$-cohomol. $\mathbb{S}$, $\mathbb{CP}$, $\mathbb{HP}$.

Question: How do we prove the Main Lemma?

The Main Lemma (K.-Wiemeler-Wilking)

Main Lemma: There exists $c(d) < 1$ such that, for any $\mathsf{T}^d \to \mathsf{SO}(V)$ with connected isotropy groups, there exists $\mathsf{S}^1 \subseteq \mathsf{T}^d$ with

$$\frac{\operatorname{codim} V^{\mathsf{S}^1}}{\dim V} \leq c(d).$$

Moreover, $c(d)$ decreases to 0 as $d \to \infty$, $c(6) = \frac{1}{3}$, and $c(9) = \frac{1}{4}$.

For $V = T_x M^n$ and T^9... there exists S^1 such that $\dim M_x^{\mathsf{S}^1} \geq \frac{3}{4} n$.

For generic representations, the ratio goes to 1 as $\dim V \to \infty$.

Looking at involutions doesn't help: $c(d)$ exists, but $\lim_{d \to \infty} c(d) = \frac{1}{2}$.

Main Lemma: Preparations

Fix a representation $\rho : \mathrm{T}^d \to \mathrm{SO}(V)$ with c.i.g.
Observation: The c.i.g. condition severely restricts the weights.
Non-example: The representation $\rho : \mathrm{T}^3 \to \mathrm{U}(8) \subseteq \mathrm{SO}(16)$ given by

$$\rho(z_1, z_2, z_3) = \mathrm{diag}(z_1, z_2, z_3, z_1^3, z_2\bar{z}_3, \underline{z_2 z_3, z_1 z_3, z_1 z_2})$$

has multiple disconnected isotropy groups:

$$\mathrm{T}^3_{e_4} \cong \mathbb{Z}_3 \times \mathrm{T}^2 \ , \quad \mathrm{T}^3_{e_5+e_6} \cong \mathrm{S}^1 \times \mathbb{Z}_2 \ , \quad \mathrm{T}^3_{\underline{e_6+e_7+e_8}} \cong \mathbb{Z}_2.$$

These are detected by the minors of the weight matrix:

$$\left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 3 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \end{array}\right].$$

Observation, formalized: If ρ has c.i.g. and weight matrix $[I|H]$, then H is totally unimodular (t.u.) (i.e., every minor is 0 or ± 1).

Main Lemma: Preparations, II

Strategy:

(1) Classify totally unimodular matrices.

(2) Solve the optimization problem.

. . . months and months of hacking away . . .

Breakthrough: t.u. matrices were classified by Seymour in 1980.

Moreover t.u. matrices modulo a good notion of equivalence correspond to abstract objects called regular matroids. Modulo equivalence, we have:

Representations ρ with c.i.g.	$\longleftrightarrow$	Totally unimodular matrices $[I\|H]$	$\longleftrightarrow$	Regular matroids $\mathcal{M}$

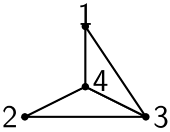
Theorem (Seymour '80): Classification of regular matroids.

Main Lemma: Preparations, III

Theorem (Seymour '80): If $\mathcal{M}$ is a regular matroid, then $\mathcal{M}$ is decomposable[1], graphic[2], co-graphic[3], or the sporadic example[4].

[1] A little more general than a direct sum of representations.

[2] $\mathcal{M}$ is derived from a graph (via the oriented incidence matrix).



$$\longleftrightarrow \quad \left[\begin{array}{ccccc} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 \\ \hline 0 & -1 & 0 & -1 & -1 \end{array}\right]$$

[3] The matroid dual is graphic. (With matrices, $[I|H] \xleftrightarrow{\text{dual}} [I|-H^T]$.)

[4] Corresponds $\rho_{\text{sporadic}} : \mathsf{T}^5 \to \mathsf{U}(10)$ with weights

$$\{e_i + e_j + e_k | 1 \le i < j < k \le 5\}.$$

Exercise: For ρ_{sporadic}, there exists $\operatorname{codim} V^{\mathsf{S}^1} \le \frac{2}{5} \dim V$.

Proof of the Main Lemma, I

We seek upper bounds on

$$c(d) = \max_{\rho} \min_{\mathrm{S}^1 \subseteq \mathrm{T}^d} \frac{\operatorname{codim} V^{\mathrm{S}^1}}{\dim V}$$

reps. ρ with c.i.g.

$$= \max_{\{h_i\},\{m_i\}} \min_{v \in \mathbb{R}^d} \sum_{\langle v, h_i \rangle \neq 0} \mu_i$$

t.u. matrices $\{h_i\} \subseteq \mathbb{Z}^d$, multiplicities $m_i = \mu_i \dim V$

$$= \max_{\mathcal{M},\mu} \min_{A} \mu(\mathcal{M} \setminus A).$$

reg. matroids $\mathcal{M}$, probability distributions μ, hyperplanes A in $\mathcal{M}$

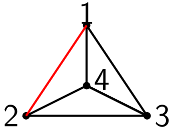
Lemma: If $\rho : \mathrm{T}^d \to \mathrm{SO}(V)$ has c.i.g. and is decomposable, then

$$c(\rho)^{-1} \geq c(d_1)^{-1} + c(d_2)^{-1} \text{ for some } d_1 + d_2 \geq d - 2.$$

Corollary: For decomposable $\rho : \mathrm{T}^9 \to \mathrm{SO}(V)$ with c.i.g., $c(\rho) \overset{**}{\leq} \frac{1}{4}$.

Proof of the Main Lemma, II: Graphic case

graph G $\longrightarrow$ weight matrix for ρ



$\longrightarrow$

$$\left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & \color{red}{1} \\ 0 & 0 & 1 & 1 & 0 & \color{red}{-1} \\ -1 & 0 & -1 & 0 & 1 & \color{red}{0} \\ \hline 0 & -1 & 0 & -1 & -1 & \color{red}{0} \end{array}\right]$$

1. Add edges to put $G \subseteq K_{d+1}$ (complete graph).
2. Extend μ by assigning $\mu_i = 0$ to the new edges.
3. Observe that $c(\rho_G) \leq c(\rho_{K_{d+1}})$.
4. Solve the problem for one graph: $c(\rho_{K_{d+1}}) = \frac{2}{d+1}$.

Proof of the Main Lemma, III: Co-graphic case

$$\begin{array}{rcl} \text{graph } G \quad \longrightarrow \quad \text{graphic matroid } \mathcal{M}[G] & \longrightarrow & \text{dual matroid } \mathcal{M}[G]^*. \\ \text{matrix } [I|H] & \longrightarrow & \text{matrix } [I|-H^T] \\ \text{hyperplanes in } \mathcal{M}[G] & \longrightarrow & \text{circuits in } \mathcal{M}[G]^* \end{array}$$

Matroid theory lets us to translate the problem into a computation of

$$c_{\mathrm{co-graphic}}(d) = \max_{(G,\mu)} \min_{\text{cycles } C} \mu(C)$$

over cubic graphs G on $3(d-1)$ edges with weights μ_i summing to 1.

Example: $G =$ Heawood graph has 21 edges and girth 6.

Constructed from 7 hexagons, where each hexagon shares an edge with every other hexagon. First draw them in a row:

Proof of Main Lemma, IV: Co-graphic case

Example: $G =$ Heawood graph has 21 edges and girth 6.

This extends to a periodic tiling:

$\Longrightarrow$ G embeds in T^2

$$\Longrightarrow \min_{\text{cycles } C} \mu(C) \leq \frac{1}{7} \sum_{\text{hexagons } C_i} \lambda(C_i) = \frac{2}{7}.$$

In fact, $c(8) = c(G) = \frac{2}{7}$, so we cannot relax the T^9 to T^8.

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