Positive curvature and fundamental group

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Fundamental groups of positively curved manifolds

Setup: (M^n, g) – compact, $\sec > 0$.

Question: What is $\pi_1(M)$?

Theorem (Bonnet-Myers): $\pi_1(M)$ finite.

Theorem (Synge): n is even $\Longrightarrow \pi_1(M) \cong 1$ or $\mathbb{Z}_2$.

Results with symmetry (Grove program):

Theorem (Wilking and Ziller, 2018):
M homogeneous $\Longrightarrow \pi_1(M) \leq \mathrm{SO}(3)$ or $\mathrm{SU}(2)$.

Theorem (Wilking, 2003):
$T^r \subseteq Isom(M^n, g)$, $r \geq \frac{n}{4} + 1 \Longrightarrow \pi_1(M)$ cyclic.

Theorem (Rong, 1999):
$S^1 \subseteq \mathrm{Isom}(M^n, g) \Longrightarrow \pi_1(M)$ has a cyclic subgroup of index $\leq w(n)$.

Fundamental groups of positively curved manifolds, II

Setup: (M^{13}, g) – compact, sec > 0.

Examples: Bazaikin Spaces, $B^{13}_{(q_1,\ldots,q_5)}$:

- Generalization of Berger's homogeneous space $B^{13} = \mathrm{SU}(5)/\mathrm{Sp}(2)\cdot S^1$.
- Infinitely many homotopy types.
- Only known examples in dimension 13 with sec > 0 (except $\mathbb{S}^{13}/\Gamma$).
- Rational cohomolgy looks like $\mathbb{C}\mathrm{P}^2 \times \mathbb{S}^9$.
- $\mathbb{Z}_3$-cohomolgy looks like either $\mathbb{C}\mathrm{P}^2 \times \mathbb{S}^9$ or $\mathbb{C}\mathrm{P}^4 \times \mathbb{S}^5$.

Theorem (K.): If $T^2 \subseteq \mathrm{Isom}(M, g)$ and $H^*(\widetilde{M}; \mathbb{Q}) \cong H^*(B^{13}_{(q_1,\ldots,q_5)}; \mathbb{Q})$, then $\pi_1(M)$ has a cyclic subgroup of index dividing 18 or 27. Moreover, if $H^*(\widetilde{M}; \mathbb{Z}_3) \cong H^*(B^{13}_{(q_1,\ldots,q_5)}; \mathbb{Z}_3)$, then the index is at most nine.

Corollary (K.): If $T^3 \subseteq \mathrm{Isom}(M, g)$ and $H^*(\widetilde{M}; \mathbb{Q}) \cong H^*(B^{13}_{(q_1,\ldots,q_5)}; \mathbb{Q})$, then $\pi_1(M)$ has a cyclic subgroup of index at most three.

Tools

Setup: (M^n, g) – compact, $\sec > 0$.

Berger fixed point theorem:
$T^k \subseteq \mathrm{Isom}(M, g) \Longrightarrow \exists T^{k-1} \subseteq T^k$ such that $M^{T^{k-1}} \neq \emptyset$.

Lemma (Davis-Weinberger, 1983):
If Γ is a group of finite order which acts freely on M^{4k+1} and trivially on $H^*(M; \mathbb{Q})$ and if $\sum_{i=0}^{2k}(-1)^i \dim H^i(M; \mathbb{Q})$ is odd, then $\Gamma \cong \mathbb{Z}_{2^a} \times \Gamma'$ where $a \geq 0$ and Γ' has odd order.

Main lemma (K.): Suppose G is a group of odd order which acts freely on M. Assume that M admits a circle action which commutes with the action of G. If $\dim H^i(M/S^1, M^{S^1}; \mathbb{Q}) \leq 1$, then for any cyclic normal subgroup $P \trianglelefteq G$, either $|G/P|$ divides $\chi(M/S^1) - \chi(M^{S^1})$ or $P \subsetneq \langle \alpha \rangle \subseteq G$.

Proof sketch

Setup: (M^{13}, g) – compact, $\sec > 0$, $H^*(\widetilde{M}; \mathbb{Q}) \cong H^*(B^{13}_{(q_1,\ldots,q_5)}; \mathbb{Q})$.

- WLOG $\widetilde{M}^{T^2} = \emptyset$.
- Choose $S^1_1 \subseteq T^2$ such that $\widetilde{M}^{S^1_1} \neq \emptyset$.
- Use equivariant cohomology to compute $H^*(\widetilde{M}^{S^1_1}; \mathbb{Q})$.
- One case: $\widetilde{M}^{S^1_1} = N_1 \sqcup N_2$, where $H^*(N_i; \mathbb{Q}) \cong H^*(\mathbb{S}^5; \mathbb{Q})$.
- Let $\Gamma \leq \pi_1(M)$ be the subgroup of index at most two which acts on N_1.
- Lemma (D-W) and Weinstein $\Longrightarrow \Gamma = \mathbb{Z}_{2^a} \times \Gamma'$ where Γ' has odd order.
- $\widetilde{M}^{T^2} = \emptyset \Longrightarrow$ there exists $S^1_2 \subseteq T^2$ such that $N_1^{S^1_2} = \emptyset$.
- Smith-Gysin sequence $\Longrightarrow \chi(N_1/S^1_2, N_1^{S^1_2}) = \chi(N_1/S^1_2, \emptyset) = 3$.
- Main lemma $\Longrightarrow \Gamma' \not\supseteq \mathbb{Z}_p \times \mathbb{Z}_p$ $(p \neq 3)$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_9 \times \mathbb{Z}_3$.
- Group theory (Burnside p-complement theorem) $\Longrightarrow$ there exists $\Gamma'' \leq \Gamma'$ of index at most three such that $\Gamma'' \not\supseteq \mathbb{Z}_p \times \mathbb{Z}_p$ for all p.
- Main lemma $\Longrightarrow$ there exists cyclic $\Gamma''' \leq \Gamma''$ of index at most three.

Proof sketch, II

Setup: (M^{13}, g) – compact, $\sec > 0$, $H^*(\widetilde{M}; \mathbb{Z}_3) \cong H^*(B^{13}_{(q_1,\dots,q_5)}; \mathbb{Z}_3)$.

Idea: rule out the possibility of $\Gamma' \supseteq \mathbb{Z}_3^2$ using spectral sequences.

- Control over $\mathbb{Z}_3$-cohomology $\Longrightarrow N_1$ is a $\mathbb{Z}_3$-cohomology $\mathbb{S}^5$ or $\mathbb{S}^2 \times \mathbb{S}^3$.

Theorem (Smith): $\mathbb{Z}_p^2$ cannot act freely on a $\mathbb{Z}_p$-cohomology sphere.

Theorem (Heller): $\mathbb{Z}_p^3$ cannot act freely on a $\mathbb{Z}_p$-cohomology $\mathbb{S}^m \times \mathbb{S}^n$. (Idea of proof: consider the Serre spectral sequence associated to the Borel fibration $G \to M_G \to BG$, where $G = \mathbb{Z}_p^3$ and M is a $\mathbb{Z}_p$-cohomology $\mathbb{S}^m \times \mathbb{S}^n$. If G acts freely, then $M_G \simeq M/G$. This, together with computing some basic bounds on the dimensions, leads to a contradiction.)

Lemma (K.): $\mathbb{Z}_p^2$ cannot act freely on a $\mathbb{Z}_p$-cohomolgy $\mathbb{S}^2 \times \mathbb{S}^3$.

Proof: Refine Heller's argument by computing the differentials.