Non-cyclic fundamental groups and the Grove symmetry program

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Positive curvature & fundamental groups

Today's focus: M – closed, positively curved Riemannian manifold.

What can we say about $\pi_1(M)$?

Classical theorems (no symmetry needed – see [Zil14] for a survey)

1. (Bonnet-Myers [Mye35, Syn35]): $\pi_1(M)$ is a finite group.
2. (Synge [Syn36]): If dim M is even, then $\pi_1(M)$ is trivial or $\mathbb{Z}_2$.

Main source of examples: Spherical space forms $\mathbb{S}^{2m-1}(1)/\Gamma$.

- Γ is trivial or $\mathbb{Z}_2 \Rightarrow$ spheres and real projective spaces.
- $\Gamma \cong \mathbb{Z}_\ell \Rightarrow$ lens spaces.
- $\Gamma \subseteq \mathsf{S}^3$ give rise to some 3-D spherical space form groups (s.s.f.g.).

Additional examples ($\Gamma = \pi_1(M)$):

- (Shankar '98) $\Gamma \subseteq \mathsf{SO}(3)$ with both homog. and inhomog. quotient.
- [Baz99, GS00] $\Gamma \cong \mathbb{Z}_6 \times \mathbb{Z}_6$ with inhomogeneous quotient.

Positive curvature, fundamental groups, & symmetry

Today's focus: M – closed, positively curved Riemannian manifold.
What can we say about $\pi_1(M)$ assuming large symmetry?

Grove symmetry program (1990s):

1. *Prove obstructions to positive curvature & large symmetry.*
2. *Discover edge cases where classifications break down.*
3. *Study those examples & use symmetry to construct examples.*

See also: Extensions to (almost) non-negative sectional curvature, positive k-Ricci curvature, positive curvature for manifolds with density,....

Theorem (see Wilking-Ziller '18): If M is homogeneous, then $\pi_1(M)$ is a finite subgroup of S^3 or $SO(3)$.

Theorem (Wilking '06): If M^n has cohomogeneity $k \leq \sqrt{n/18} - 1$, then $\pi_1(M^n)$ is isomorphic to a spherical space form group.

Motivating the cyclic conjecture

Setup: $(M^{2m-1}, \sec > 0)$ – closed with T^r symmetry (e.g., $\mathbb{S}^{2r-1}/\mathbb{Z}_\ell$)

Theorem (Grove-Searle '94): $r \leq m$.

Theorem (Rong '05, Theorem D): If $\pi_1(M)$ is not cyclic, then $r \leq \frac{m}{2}$.

(If $2 \nmid m$, then $r < \frac{m}{2}$.)

Theorem (Frank-Rong-Wang '13): If $\pi_1(M)$ is not cyclic, then $r \leq \frac{m}{3}$ whenever 2 does not divide m.

(If $2 \nmid m$ and $3 \nmid m$, then $r < \frac{m}{3}$.)

Conjecture (Wang '10, K. '17): If $\pi_1(M)$ is non-cyclic, then $r \leq \frac{m}{p}$ where p is the minimum prime dividing m.

Models: Spherical space forms $\mathbb{S}^{2pr-1}/\Gamma$ with T^r-symmetry (K. '17).

Evidence for the cyclic conjecture

Setup: $(M^{2m-1}, \sec > 0)$ – closed with T^r symmetry.

Define: p is the minimum prime dividing m.

Cyclic conjecture: If $r > \frac{m}{p}$, then $\pi_1(M)$ is cyclic.

- True if $p = 2$ (Rong) or $p = 3$ (Frank-Rong-Wang).
- True if $\tilde{M}$ is a homology sphere (Wang '10, K. '17).
- True if the T^r-action has no fixed point and $m \geq m(p)$ (K. '17)

Theorem (K.-Wang): If $p = 5$ in the conjecture and $m \neq 5$, then

1. $\pi_1(M)$ is cyclic, or
2. $\pi_1(M)$ has a cyclic subgroup of index three and $m \equiv 2 \bmod 3$.

Hardest case: The 49-dimensional base case(!). It relies on [Ron05, Ken17].

Confirms the conjecture for a new infinite family of dimensions, $49 + 60k$.

Almost cyclic fundamental groups, I

Setup: M^n – closed, Riemannian manifold with $\sec > 0$.

Theorem (Rong's Almost Cyclic Theorem '99 & '05):

$$\mathsf{S}^1\text{-symmetry implies } \pi_1(M^n) \text{ is } w(n)\text{-cyclic.}$$

As a corollary,

$$\mathsf{T}^r\text{-symmetry implies } \pi_1(M^n) \text{ is } w_r(n)\text{-cyclic}$$

for some $w_r(n)$ such that $w_r(n) \searrow 1$ as $r \nearrow \frac{n+1}{4}$.

Problem: Estimate $w_r(n)$.

Theorem (K. '17 Theorem 8.2 + Khalili Samani '20): For $n \equiv 1(4)$,

1. $r \geq \frac{n}{12} + 3$ implies $w_r(n) \leq 5$.
2. $r \geq \frac{n}{20} + 5$ implies $w_r(n) \leq 27$.

Questions: Does $r \geq \delta n$ imply $w_r(n) \leq c(\delta)$? Is $n \equiv 1 \bmod 4$ necessary?

Almost cyclic fundamental groups, II

Theorem (Rong '99): T^2-symmetry implies $\pi_1(M^n)$ is $w_2(n)$-cyclic.

Theorem (K.-Khalili Samani): If T^2 acts on M^7 and $\tilde{M} \sim_{\mathbb{Q}} \mathbb{S}^7$ or Eschenburg space, then $\pi_1(M)$ has a cyclic subgroup of index ≤ 12.

Example: The "index 12" is sharp in both cases (see [Sha98, WZ18]):

1. T^2 acts on $\mathbb{S}^7/2I$, where $2I \subseteq \mathsf{S}^3$ is the binary icosohedral group.
2. T^2 acts on E^7/I, where $I = A_5 \subseteq \mathsf{SO}(3)$ is the icosohedral group.

Theorem (K. '17): If T^2 acts and $\tilde{M} \sim_{\mathbb{Q}} \mathbb{S}^{13}$, then $\pi_1(M)$ is cyclic.

Theorem (Khalili Samani '20): If T^2 acts and $\tilde{M}^{13} \sim_{\mathbb{Z}}$ Bazaikin space, then $\pi_1(M)$ has a cyclic subgroup of index $1, 2, 3, 6$, or 9.

Question: Do Bazaikin spaces admit free actions by any non-cyclic group?

Tools, I: Transformation groups & $\sec > 0$

1. Theory of transformation groups ($\sec > 0 \Rightarrow \mathsf{T}^{r-1}$ has a fixed point).

2. Induction & error correcting codes (see Wilking '03)

$\Rightarrow$ may assume the existence of fixed point sets with small codimension.

3. Connectedness lemma (Wilking '03) implies *periodicity in cohomology*.

4. Steenrod powers and secondary cohomology operations (K. '13, K. '17):

- If $\dim M = 4k+1$, then $\tilde{M}$ is a rational homology sphere.
- If $\tilde{M}$ is $(2p)$-periodic, then $\tilde{M}$ is a $\mathbb{Z}_p$-homology sphere.

Tools, II: Group-theoretic restrictions on $\pi = \pi_1(M)$

1. (Davis-Weinberger '83): If $\tilde{M} \simeq \mathbb{S}^{4k+1}$ and sec > 0, then $\pi \cong \mathbb{Z}_{2^e} \times \Gamma$.

2. ([FRW13]): If π acts freely on $(M^n, \sec > 0)$ and on a totally geodesic N^{n-2}, then π has 2-periodic group cohomology and hence is cyclic.

3. (Smith 1930s and K. '17): If $\tilde{M}$ has $(2p)$-periodic $\mathbb{Z}_p$-cohomology, then $\tilde{M}$ is a $\mathbb{Z}_p$-homology sphere and $\mathbb{Z}_p \times \mathbb{Z}_p$ does not act freely on M.

Main Lemma (Khalili Samani, [Kha]): Let M be a closed, positively curved manifold, and assume that M has S^1-symmetry that commutes with a free, *homologically trivial* action by an odd-order group Γ.

For any maximal cyclic subgroup $\langle \alpha \rangle \subseteq \Gamma$, the order of $N(\langle \alpha \rangle)/\langle \alpha \rangle$ divides $\chi(M/\mathsf{S}^1) - \chi(M^{\mathsf{S}^1})$.

Generalizes obstructions in [Ken17] and Sun-Wang [SW09].

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