The Halperin Conjecture in Small Formal Dimensions

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Definition

Definition (Graded Polynomial Algebra)

$\mathbb{Q}[x_1, x_2, \ldots, x_n]$ endowed with a grading $|x_j|$

Definition (Regular Sequence)

Polynomials $u_1, \ldots, u_n$ form a **regular sequence** if each u_i is not a zero divisor in quotient $\mathbb{Q}[x_1, \ldots, x_n]/(u_1, \ldots, u_{i-1})$

Example: $\mathbb{Q}[x_1, x_2]/(x_1^2 - x_2^2, x_1 x_2)$
Non-example: $\mathbb{Q}[x_1, x_2]/(x_1^2, x_1 x_2)$ since $(x_1 x_2)^2 = x_1^2 x_2^2$

Definition of PEA: Algebraic Version

Definition (Positively Elliptic Algebras H^*)

$H^* = \mathbb{Q}[x_1, \ldots, x_k]/(u_1, \ldots, u_k)$ is a **PEA** if (1) each u_i is a **homogeneous** polynomial, (2) polynomials $u_1, \ldots, u_k$ form a regular sequence, and (3) each $|x_i|$ is a positive **even** integer.

Example

$H^* = \mathbb{Q}[x_1, x_2]/(x_1^2 - x_2, x_2^2)$ graded by $|x_1| = 2, |x_2| = 4$ is vector space spanned by $1, x_1, x_1^2, x_1^3$. (Isomorphic to $\mathbb{Q}[x_1]/(x_1^4)$)

Property (Equivalent conditions)

The following properties are equivalent for a PEA:

(a) Polynomials $u_1, \ldots, u_k$ form a regular sequence

(b) $\mathrm{Krull\,dim}\, H^* = 0$

(c) H^* has finite dimension as a vector space

(d) ***Formal dimension*** $\mathrm{fd} H^* = \sum_i |u_i| - |x_i|$ *is finite*

Algebraic Version: Pure Model

Recall that $\mathbb{Q}[x_1, x_2]/(x_1^2 - x_2, x_2^2) \cong \mathbb{Q}[x_1]/(x_1^4)$

Theorem (Pure Model)

Given a positively elliptic algebra H^*, there exists variables x_i, positive and even degrees $|x_i|$, and homogeneous polynomials

$$u_i \in \mathbb{Q}^{\geq \mathbf{2}}[x_1, \ldots, x_k] = \operatorname{span}\{x_1^{a_1} \cdots x_k^{a_k} | a_1 + \cdots + a_k \geq 2\}$$

such that $H^* \cong \mathbb{Q}[x_1, \ldots, x_k]/(u_1, \ldots, u_k)$.
Moreover, we can choose a nice representative that satisfying
(1) $|x_1| \leq \cdots \leq |x_k|$
(2) $|u_1| \leq \cdots \leq |u_k|$
(3) $|u_i| \geq 2|x_i|$
(4) $\mathrm{fd} H^* = \sum_{i=1}^{k} |u_i| - |x_i|$
(5) Strong Algebraic Condition (SAC)

Definition: Topological Version

Definition (Positively Elliptic Space M)

Rational Homotopy Group $\pi_*(M) \otimes \mathbb{Q}$ where $\pi_*(M)$ is homotopy group.
An oriented, closed manifold is (rationally) elliptic if $\dim \pi_*(M) \otimes \mathbb{Q} < \infty$.
An elliptic manifold is positively elliptic if $\chi(M) > 0$.

Theorem (Connection with Topology)

*The rational cohomology algebra $H^*(M; \mathbb{Q})$ of **Positively Elliptic Space** M is a **Positively Elliptic Algebra**.*

Property (Formal Dimension)

The dimension of manifold M equals the formal dimension $\mathrm{fd}(H^*)$ $= \sum_i |u_i| - |x_i|$, which is also the greatest integer d such that $H^d \neq 0$.

More on Rational Homotopy Theory

Yves Felix, Stephen Halperin, & Jean-Claude Thomas (2000)
Rational Homotopy Theory
Graduate Texts in Mathematics

Chapter 32 is relevant to positively elliptic algebras H^*.

Chapter 39 lists seventeen open problems in this area.
The first is Halperin's Conjecture.

Halperin's Conjecture: Algebraic Version

In 1982, Meier reformulated the Halperin Conjecture entirely algebraically:

Conjecture (Algebraic Halperin Conjecture)

PEAs do not admit non-trivial **derivations** of **negative** degree.

Definition (Derivation)

Derivation on H^* is a linear map $\delta : H^* \to H^*$ that satisfies
(i) changing degree by some integer $|\delta|$. i.e. $\delta(H^n) \subseteq H^{n+|\delta|}$
(ii) Leibniz's Rule: $\delta(ab) = \delta(a)b + (-1)^{|\delta||a|}a\delta(b)$ for homogeneous a, b

More on Derivation

Example (PEA can admit $|\delta| > 0$)

$H^* = \mathbb{Q}[x_1, x_2]/(x_1^2 + x_2^2, x_1x_2)$ where $|x_1| = |x_2| = 2$.
Let $\delta(x_1) = x_1^2$ and $\delta(x_2) = 0$.
This is a derivation on $\mathbb{Q}[x_1, x_2]$ with degree $+2$.
δ descends to a derivation on H^* because
$\delta(x_1^2 + x_2^2), \delta(x_1x_2) \in (x_1^2 + x_2^2, x_1x_2)$

More on Derivation

Example (Non-PEA can admit $|\delta| < 0$)

$H^* = \mathbb{Q}[x_1, x_2]/(x_1^2, x_1x_2)$ where $|x_1| = 2, |x_2| = 4$.
Let $\delta(x_1) = 0$ and $\delta(x_2) = x_1$.
This is a derivation on $\mathbb{Q}[x_1, x_2]$ with degree -2.
δ descends to a derivation on H^* because $\delta(x_1^2), \delta(x_1x_2) \in (x_1^2, x_1x_2)$

Example (Non-PEA can admit $|\delta| < 0$)

Let $H^* = \mathbb{Q}[x_2, x_3]/(x_2^2, x_3^2)$ where $|x_2| = 2$ and $|x_3| = 3$.
Let $\delta(x_2) = 0$ and $\delta(x_3) = x_2$.
This is a derivation on $\mathbb{Q}[x_2, x_3]$ with degree -1
δ descends to a derivation on H^* because $\delta(x_2^2) = 0$ and $\delta(x_3^2) = 0$.
(Leibniz's Rule yields -1 here)

AHC: one simple case

Conjecture (Algebraic Halperin Conjecture)

PEAs do not admit non-trivial derivations of negative degree.

Example (One Simple Case)

Consider $H^* = \mathbb{Q}[x_1, x_2]/(x_1^2 + x_2^2, x_1 x_2)$ with $|x_1| = |x_2| = 2$.
Suppose derivation on H^* with $|\delta| < 0$, then $\delta(x_1), \delta(x_2) \in \mathbb{Q}$.
Notice $0 = \delta(0) = \delta(x_1 x_2) = \delta(x_1)x_2 + x_1\delta(x_2) \in H^2$.
Linear independence implies that $\delta(x_1) = \delta(x_2) = 0$.

Lemma (Land-in-Zero Lemma, [AK 20])

If $\delta(x) \in H^0 \cong \mathbb{Q}$ then $\delta(x) = 0$.

Main Theorem

Theorem (Kennard-W.)

Algebraic Halperin Conjecture holds for formal dimensions up to 20.

Recall that $H^* = \mathbb{Q}[x_1, \ldots, x_k]/(u_1, \ldots, u_k)$ and $\mathrm{fd} H^* = \sum_{j=1}^k |u_j| - |x_j|$

Theorem (Previous Work)

The Algebraic Halperin Conjecture holds for the following situations:

1. *[Lupton '82, Chen '99] the number of generators is* $k = 1, 2, 3$.
2. *[Papadima-Paunescu '96] for "Reduced Case"*
3. *[Amann-Kennard '20] for* $\mathrm{fd} H^* \leq 16$

Strategy from Previous Work

Definition (Degree Type)

For PEA $H^* = \mathbb{Q}[x_1, \ldots, x_k]/(u_1, \ldots, u_k)$, we write its **Degree Type**

$$(|x_1|, \ldots, |x_k|; |u_1|, \ldots, |u_k|)$$

Definition (Split PEA)

A Positively Elliptic Algebra H^* splits if we have a short exact sequence

$$0 \to K^* \to H^* \to Q^* \to 0$$

where K^* and Q^* are non-trivial PEA.

For example, pure model H^* **splits** if there exists $l < k$ such that relations $u_1, \ldots, u_l$ only depend on $x_1, \ldots, x_l$.

Strategy from Previous Work

Theorem (Markl's Theorem)

Let H^* be a PEA with a non-zero derivation of negative degree.
If H^* splits as $0 \to K^* \to H^* \to Q^* \to 0$,
then K^* or Q^* also has a non-zero derivation of negative degree.

This gives a smaller PEA: smaller k, fd, and dim.
Also we can assume PEA does not split in the induction step.

Tools in Amann-Kennard's Work

Lemma (Land-in-Zero Lemma)

Let δ be a derivation of negative degree on H^*. If there exists $x \in H^i$ for some $i > 0$ such that $\delta(x) \in H^0$, then $\delta(x) = 0$.

Lemma ($k - 1$ Lemma)

If δ is a derivation of negative degree on H^* such that $\delta(x_i) = 0$ for $k - 1$ of the generators x_i, then $\delta = 0$.

Lemma (Degree Inequality)

Assume that PEA $H^* = \mathbb{Q}[x_1, \ldots, x_k]/(u_1, \ldots, u_k)$ does ***NOT*** *split, then:*

1. *If $i < k$, then $|u_i| \geq |x_1| + |x_{i+1}|$.*
2. *If $\delta(x_2) = \lambda x_1^\alpha$ for some non-zero $\lambda \in \mathbb{Q}$, then $|u_1| \geq |x_1| + |x_3|$.*

1st New Tool: Large Relations Lemma

Lemma (Large Relations Lemma)

(1) Assume that PEA $H^* \cong \mathbb{Q}[x_1, \ldots, x_k]/(u_1, \ldots, u_k)$ *does not split.*
(2) Assume that H^* *admits a derivation* δ *of negative degree such that the map* $\delta : H^4 \to H^2$ *has rank* $m \geq 1$.
Let $g_i = \#$ *generators with degree* i, *and* $r_j = \#$ *relations with degree* j.
(3) If $g_6 + g_{10} + g_{14} + \ldots = 0$, *then*

$$r_{12} + r_{16} + \ldots \geq (k - g_2 - g_4) + 1$$

In particular, $|u_k| \geq 12$.

Proof of the Large Relations Lemma (Set-up)

Land-in-Zero Lemma $\Rightarrow \delta(x_i) = 0$ for $1 \leq i \leq g_2$. After changing basis,

$$\delta(x_{g_2+i}) = \begin{cases} x_i & \text{if} \quad 1 \leq i \leq m \\ 0 & \text{if} \quad m < i \leq g_4 \end{cases}$$

Consider $\{u_j | \ j \in J\}$ relations with degree 8. Decompose each as

$$u_j = p_j(x_{g_2+1}, \ldots, x_{g_2+m}) + r_j$$

where p_j is a quadratic polynomial and where r_j is in the ideal

$$I_0 = (x_1, \ldots, x_{g_2}) + (x_{g_2+m+1}, \ldots, x_{g_2+g_4}).$$

Fix $J' \subseteq J$ s.t. $\{p_j | j \in J'\}$ is a basis for $\mathrm{span}\{p_j | \ j \in J\}$.

(continued)

Proof of the Large Relations Lemma (Case 1)

Case 1: $|J'| \geq m$

Degree reasons $\Rightarrow \delta^2(r_j) = 0$ for all $j \in J$. Therefore

$$\delta^2(u_j) = 2p_j(x_1, \ldots, x_m)$$

Notice that $\delta^2(u_j)$ has degree 4 and $\delta^2(u_j) \in (u_1, \ldots, u_k)$, so

$$p_j(x_1, \ldots, x_m) \in \mathrm{span}\{u_i : |u_i| = 4\}$$

Linear independence of $\{p_j |\ j \in J'\} \Rightarrow$

WLOG, $u_1, \ldots, u_m$ only depends on $x_1, \ldots, x_m$

But this means that H^* splits. Contradiction!

(continued)

Proof of the Large Relations Lemma (Case 2)

Case 2: $|J'| \leq m - 1$

Choice of $J' \Rightarrow$ WLOG, $p_j = 0$ for $j \in J \setminus J'$.

Consider the ideal

$$I = I_0 + \left(\{u_j |\ j \in J'\}\right) + \left(\{u_j |\ |u_j| \in \{12, 16, \ldots\}\right).$$

Degree reasons & choice of $J' \Rightarrow$ all relations $u_j \in I$.

Hence H^* projects onto $\mathbb{Q}[x_1, \ldots, x_k]/I$.

Since H^* is finite-dimensional, I must have at least k generators. Therefore

$$(g_2 + g_4 - m) + (m - 1) + (r_{12} + r_{16} + \ldots) \geq k$$

□

2nd New Tool: Top-to-Bottom Lemma

Lemma (Top-to-Bottom Lemma)

(1) Assume that PEA $H^* \cong \mathbb{Q}[x_1, \dots, x_k]/(u_1, \dots, u_k)$ *does not split.*
(2) Assume that $|u_k| < 3|x_k|$*.*
(3) If there exists a derivation δ *with negative degree and* $\exists l \geq 1$ *s.t.*

$$\delta^l \colon H^{|x_k|} \to H^{|x_1|}$$

exists and has rank at least 1,
then this map has rank at least 2.

Reminiscent of $k-1$ Lemma: $\mathrm{rank}\delta \geq 1 \Rightarrow \mathrm{rank}\delta \geq 2$.

Proof of Top-to-Bottom Lemma (Set-up)

Assume that $|\delta|$ divides $|x_k| - |x_1|$, and fix $l \geq 1$ s.t. $\delta^l : H^{|x_k|} \to H^{|x_1|}$.
Suppose $\mathrm{rank}\delta^l = 1$, then WLOG,

$$\delta^l(x_k) = x_1 \text{ and } \delta^l(x_i) = 0 \text{ for } i < k$$

Consider ideal $I = (x_1, \ldots, x_{k-1})$.
Finite-dimensionality of $H^* \Rightarrow$ there exists u_i not in I.
Since $|u_i| < 3|x_k|$, we must have

$$u_i = \lambda x_k^2 + x_k f + g$$

for some non-zero $\lambda \in \mathbb{Q}$ and some $f, g \in \mathbb{Q}[x_1, \ldots, x_{k-1}]$.
By scaling u_i s.t. $\lambda = 1$. Replacing x_k by $x_k + \frac{1}{2}f$, we may assume that

$$u_i = x_k^2 + g'$$

(continued)

Proof of Top-to-Bottom Lemma (Case 1)

We apply δ^{2l} to equation $u_i = x_k^2 + g'$.

Case 1: $\delta^{2l}(g') = 0$.
RHS:

$$\delta^{2l}(x_k^2) = \binom{2l}{l} \left(\delta^l x_k\right)^2 = \binom{2l}{l} x_1^2.$$

LHS: $\delta^{2l}(u_i) \in (u_1, \ldots, u_k)$ with (minimal) degree $2|x_1|$.

Then $\delta^{2l}(u_i) \in \mathrm{span}\{u_j : |u_j| = 2|x_1|\}$. WLOG, $\delta^{2l}(u_i) = u_1'$.
We have $u_1' \in \mathbb{Q}[x_1]$ so H^* splits. Contradiction!

(continued)

Proof of Top-to-Bottom Lemma (Case 2)

Case 2: $\delta^{2l}(g') \neq 0$.
$g' \in \mathbb{Q}[x_1, \ldots, x_{k-1}]$ has a monomial $x_{i_1} \cdots x_{i_p}$ such that

$$\delta^{2l}(x_{i_1} \cdots x_{i_p}) \neq 0.$$

By the Leibniz's Rule, there exists $j_1 + \cdots + j_p = 2l$ such that

$$\delta^{j_1}(x_{i_1}) \cdots \delta^{j_p}(x_{i_p}) \neq 0$$

Each term has degree at least $|x_1|$. Summing, we have

$$p|x_1| \leq |\delta^{j_1}(x_{i_1}) \cdots \delta^{j_p}(x_{i_p})| = |\delta^{2l}(g')| = 2|x_1|.$$

Hence $p \leq 2$. Pure model $\Rightarrow p = 2$ here.
Thus there exists some $\delta^l(x_i) \neq 0$ with $x_i \neq x_k$, contradiction.

$\square$

Proof Sketch: Part of Proposition 4.2

Proposition (4.2)

Let H^* *be a PEA that does not split. If there exists a non-zero derivation of negative degree and* $|x_{k-1}| + |x_k| = 10$*, then either* $\mathrm{fd}\, H^* > 20$ *or the degree type is equal to*

$$(2, 2, 2, 4, 6; 4, 4, 6, 10, 12).$$

Proof Sketch: Proposition 4.2

Proof.

Degree reasons & $k-1$ Lemma $\Rightarrow$ WLOG, $|x_{k-1}| = 4$ and $|x_k| = 6$.

SAC(6)$\Rightarrow$ only need to consider $|u_k| = 12$ if $\mathrm{fd} H^* \leq 20$.

Top-to-Bottom Lemma $\Rightarrow$ $\delta^2(x_k) = 0$.

$k-1$ Lemma $\Rightarrow$ $\delta(x_{k-1}) \neq 0$. WLOG, $\delta(x_{k-1}) = x_1$.

$\delta(x_k) = p(x_2, \ldots, x_{k-2})$ if we replace x_k by $x_k - l x_{k-1}$.

Therefore, $k \geq 4$.

Consider $k \geq 5$, Degree Inequality: $\mathrm{fd} H^* \geq (k-3)|x_1| + 4 + 6 + 6 \geq 20$.

Degree Types is $(2, |x_2|, |x_3|, 4, 6; 2 + |x_2|, 2 + |x_3|, 6, 10, 12)$.

Pure Model: $|u_j| \geq 2|x_j|$ so $|x_2| = |x_3| = 2$.

Other techniques to show contradiction when $k = 4$. □

Main Theorem: Proof Sketch

$\lvert x_{k-1}\rvert + \lvert x_k\rvert$	# exceptional cases
≤ 8	3
$= 10$	1
≥ 12	$2^\dagger$

Note:

1. All exceptional cases have formal dimensional 18 or 20.

$2^\dagger$. One case with degree type $(2, 2, 6, 6; 4, 8, 12, 12)$ is difficult.

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