Torus actions and positive curvature

Lee Kennard
Syracuse University

Joint with Michael Wiemeler (Münster) and Burkhard Wilking (Münster)

Virtual seminar on geometry with symmetries

6 May 2020
Positive curvature and symmetry

Setup: $(M, g)$ – closed, orientable, $\sec > 0$ (e.g., $S^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{O}P^2$).

(Other known examples in $\dim M \in \{6, 7, 12, 13, 24\}$)

**Conjecture** (Hopf 1930s): $\sec > 0 \implies M \neq S^2 \times S^2$.

**Conjecture** (Hopf 1930s): $\sec > 0 \implies \chi(M^{2n}) > 0$.

Grove symmetry program (1990s): Study $\sec > 0$ with symmetry.

(Homogeneous spaces, cohomogeneity one manifolds, quotients, . . . .)

Major developments:

- (Shankar ’98) Resolution of Chern’s 1965 question on $\pi_1(M)$.
- (Dearricott ’11, Grove-Verdiani-Ziller ’11) New example with $\sec > 0$.
- (Gromoll-Meyer ’74, Grove-Ziller ’00, Goette-Kerin-Shankar ’20)
  Many 2-connected 7-manifolds admit $\sec \geq 0$, including all exotic $S^7$. 
Positive curvature and torus symmetry

Setup: \((M^{2n}, g)\) – closed, orientable, sec > 0 (e.g., \(S^{2n}, \mathbb{C}P^n, \mathbb{H}P^n, \mathbb{O}P^2\)).

**Conjecture** (Hopf 1930s): \(\sec > 0 \implies \chi(M^{2n}) > 0\).

Today’s symmetry assumption: \(T^d\) acts isometrically with \(d \geq C(n)\).

**Theorem** (Grove-Searle ’94): \(C(n) = n \implies\) diffeomorphism rigidity.

**Theorem** (Wilking ’03): \(C(n) = \frac{n}{2} + 1 \implies\) homotopy rigidity.

**Theorem** (Amann-K. ’14): \(C(n) = 2 \log_2(2n), b_3 = 0 \implies \chi(M) > 0\).

**Theorem** (K.-Wiemeler-Wilking): \(C(n) = \log_2(2n) \implies \chi(M)\) standard.

**Theorem** (K.-Wiemeler-Wilking): \(C(n) = 5 \implies \chi(M) > 0\).

Compare: (Rong ’99), (Dessai ’05 & ’07), (Weisskopf ’17).
Tools, I: Existing tools and eliminating \( b_3 = 0 \)

Setup: \((M^{2n}, g)\) – closed, orientable, sec > 0, invariant under \( T^d \).

General structure theory

(General theory of transformation groups + classical results from sec > 0.)

+ Connectedness & periodicity lemmas (Wilking ’03)

(Morse theory of geodesics + second variation of energy)

+ Rational four-periodicity theorem (K. ’13)

(Steenrod squares and Steenrod powers; implies 4-periodic Betti numbers)

+ \( b_3 \) lemma (K.-Wiemeler-Wilking)

(Equivariant cohomology & global analysis of fixed point set)

⇓ Bake at 350\(^\circ\)F

\[
\text{For } F \subseteq M^{T^d}, \text{ it suffices}^* \text{ to find } T^d \supseteq H_i \supseteq H \text{ with } H_i/H \cong S^1 \text{ such that the fixed point components } F \subseteq N_i \subseteq N \text{ satisfy } N_1 \cap N_2.
\]
Tools, II: Reducing $C(n)$ to 5

**Question:** How do we get transverse intersections?

Fix $p \in F \subseteq M^{T^d}$, and study isotropy representation $\rho : T^d \to \text{SO}(V)$. We need to find $T^d \supseteq H_i \supseteq H$ such that $V^H_1 \cap V^H_2$ in $V^H$.

(If $C(n) \approx \log_2 n$, one can look at subgroups of $\mathbb{Z}_2^d \subseteq T^d$.)

**S$^1$-splitting (K.-Wiemeler-Wilking):** If $\rho : T^d \to \text{SO}(V)$ is faithful and $d \geq 3$, there exists $H \subseteq T^d$ such that the induced representation $T^{d-1} = T^d / H \to \text{SO}(V^H)$ factors through a product representation $S^1 \times T^{d-2} \to \text{SO}(V_1) \times \text{SO}(V_2)$ for some decomposition $V^H = V_1 \oplus V_2$.

In $V^H$, the fixed point sets of $S^1$ and $T^{d-2}$ intersect transversely.

* If $d \geq 5$, one can iterate to gain control over dimensions of $V^{H_i} \subseteq V^H$. 
**S¹-splitting**: Given \( \rho : T^d \to SO(V) \) with \( d \geq 3 \), there exists \( H \subseteq T^d \) such that the induced representation \( T^{d-1} = T^d / H \to SO(V^H) \) factors through \( S¹ \times T^{d-2} \to SO(V_1) \times SO(V_2) \), for some \( V^H = V_1 \oplus V_2 \).

**Example 1:**
\[
\rho(z_1, z_2, z_3) = \text{diag}(z_1, z_2, z_3, z_1 \bar{z}_2, z_2 \bar{z}_3, z_1 \bar{z}_3) \in U(6).
\]
Take \( H = \{(z, z, 1)\} \) \( \sim \) \( V^H = C_{e3} \oplus C_{e4} \) and \( \bar{\rho} \) splits.

**Example 2:**
\[
\rho(a, b, c) = \text{diag}(a, b, c, ab, ac, bc, a\bar{b}, a\bar{c}, b\bar{c}, ab\bar{c}, a\bar{b}c, \bar{a}bc).
\]
First look at \( \mathbb{Z}_2 \cong \langle (-1, -1, -1) \rangle \) \( \sim \) \( V^{\mathbb{Z}_2} = C_{e4} \oplus \ldots \oplus C_{e9} \).
Then look at \( H = \mathbb{Z}_2 \cdot \{(1, 1, z)\} \). \( \sim \) \( V^H = C_{e4} \oplus C_{e7} \). \( \sim \) \( \bar{\rho} \) splits

**Observation:** The splitting holds if there exists a finite isotropy group \( F \).
(Look at induced representation \( T^d \cong T^d / F \to SO(V^F) \). Use induction.)

**Reduction:** In the proof, we may assume connected isotropy groups (c.i.g.)
**S¹-splitting, II: Torus representations with c.i.g.**

**Setup:** Assume $\rho: T^d \to \text{SO}(V)$ has **connected isotropy groups**.

**Non-example:** The representation $\rho: T^3 \to \text{U}(6)$ given by

$$\rho(z_1, z_2, z_3) = \text{diag}(z_1, z_2^3, z_2z_3, z_1z_3, z_1z_2)$$

has disconnected isotropy groups

$$T^3_{e_2} \cong \mathbb{Z}_3 \times T^2, \quad T^3_{e_3+e_4} \cong S^1 \times \mathbb{Z}_2, \quad T^3_{e_4+e_5+e_6} \cong \mathbb{Z}_2.$$  

**Example 1:** $\rho: T^4 \to \text{U}(10) \subseteq \text{SO}(20)$ given by

$$\text{diag}(z_1, z_2, z_3, z_4, z_1\bar{z}_2, z_1\bar{z}_3, z_1\bar{z}_4, z_2\bar{z}_3, z_2\bar{z}_4, z_3\bar{z}_4).$$

**Example 2:** $\rho: T^4 \to \text{U}(9) \subseteq \text{SO}(18)$ given by

$$\text{diag}(z_1, z_2, z_3, z_4, z_1\bar{z}_3, z_1\bar{z}_4, z_2\bar{z}_3, z_2\bar{z}_4, z_1z_2\bar{z}_3\bar{z}_4).$$

**Classification for $d = 4$:** Any torus representation with **c.i.g.** is equivalent to a subrepresentation of one of these examples (ignoring multiplicities).
Setup: Assume $\rho : T^d \hookrightarrow \text{SO}(V)$ has connected isotropy groups.

Initial analysis:

Build a matrix $H$ whose columns are weights.

Lemma: c.i.g. $\iff$ every $d \times d$ submatrix $H'$ has $\det(H') = 0$ or $\pm 1$.

For example: $\rho(z, w) = (z, w, zw, \bar{z}w)$ has $H = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

Submatrix $H'$ has $\det = 2$, corresponding to

$$T^2_{e_3 + e_4} = \ker(zw) \cap \ker(\bar{z}w) = \langle (-1, -1) \rangle \cong \mathbb{Z}_2.$$ 

We may assume the $e_i$ are weights, so every submatrix has $\det = 0, \pm 1$.

In particular, the weights $h_i \in \{-1, 0, 1\}^d$ and $\# \{ h_i \} \leq 3^d$. 

Lee Kennard (Syracuse) 
Torus actions and positive curvature 
6 May 2020 7 / 10
Further combinatorial analysis:

(1) $\# \text{ weights } \leq \frac{d(d+1)}{2}$, and this is sharp. Independent of dim $V$!

(2) There is a basis $h_1, \ldots, h_d$ of weights such that $h_1$ is the one and only weight in the affine subspace $h_1 + \langle h_3, \ldots, h_d \rangle$. (Implies the $S^1$-splitting.)

**Example 1:** $(z_1, z_2, z_3, z_4, z_1\bar{z}_2, z_1\bar{z}_3, z_1\bar{z}_4, z_2\bar{z}_3, z_2\bar{z}_4, z_3\bar{z}_4)$ has 10 weights. $e_1$ is the only weight in $e_1 + \langle e_2 - e_3, e_3 - e_4 \rangle$, so $H = \{(1, z, z, z)\}$ gives rise to a splitting $\bar{\rho} : T^3 \to U(4)$ of the form $\bar{\rho}(x, y, z) = \text{diag}(x, y, z, yz)$.

**Example 2:** $(z_1, z_2, z_3, z_4, z_1\bar{z}_3, z_1\bar{z}_4, z_2\bar{z}_3, z_2\bar{z}_4, z_1z_2\bar{z}_3\bar{z}_4)$. Only one weight in $e_1 + e_2 - e_3 - e_4 + \langle e_2, e_3 \rangle$, so take $H = \{ (\bar{z}, 1, 1, z) \}$. 
Proof of main theorem

Setup: \((M^{2n}, g)\) is closed, oriented, positively curved with \(T^5\) symmetry.

Apply \(S^1\)-splitting twice to the isotropy representation at \(p \in F \subseteq M^{T^5}\):

There exist \(H^2 \subseteq T^5\) and three circles \(S^1_i \subseteq T^5/H\) such that \(N_{p_i} \cap N_{p_j}\).

4-periodicity + \(b_3\) lemma \(\Rightarrow N = M_p^H \cong \mathbb{Q} S^m, \mathbb{C}P^m, \mathbb{H}P^m, S^2 \times \mathbb{H}P^m\).

Localization theorem \(\Rightarrow F \cong \mathbb{Q} S^l, \mathbb{C}P^l, \mathbb{H}P^l, S^2 \times \mathbb{H}P^l, S^2 \times \mathbb{C}P^l\).

- \(F \not\cong \mathbb{Q} S^2 \times \mathbb{C}P^l\) is easy (cohomology is not periodic).
- \(F \not\cong \mathbb{Q} S^2 \times \mathbb{H}P^l\) is hard (global analysis of \(M^{T^5}\) and isotropy weights).
Theorem 1: If $T^5$ acts on a closed, orientable, positively curved $M^{2n}$, then every fixed point component of $T^5$ is a rational $S^k$, $\mathbb{CP}^k$, or $\mathbb{HP}^k$.

Why work so hard for the cohomology of $M^{T^5}$?

Theorem 2: If $M^n$ (closed, orientable) admits an equivalently formal $T^8$-action such that every fixed point component of every $T^5 \subseteq T^8$ is a rational $S$, $\mathbb{CP}$, or $\mathbb{HP}$, then $M$ is a rational $S^n$, $\mathbb{CP}^n$, or $\mathbb{HP}^n$.

- Partial converse to (Smith ’38, Bredon ’64): Fixed point components of torus actions on $S$, $\mathbb{CP}$, $\mathbb{HP}$ are again $S$, $\mathbb{CP}$, $\mathbb{HP}$.
- Special case (GKM action): $\dim(M^{T^8}) = 0$ and every $\dim(M^{T^7}) \leq 2$. We use results from Goertsches-Wiemeler ’15.

Corollary: If $T^8$ acts on a closed, orientable, positively curved $M^{2n}$ with $H^{\text{odd}}(M; \mathbb{Q}) = 0$, then $M$ is a $\mathbb{Q}$-cohomology $S^{2n}$, $\mathbb{CP}^n$, or $\mathbb{HP}^n$. 
Ongoing work

Question 1: Do we need the assumption $H_{\text{odd}}(M; \mathbb{Q}) = 0$ in the $T^8$ result?

- The Bott-Grove-Halperin conjecture + the $T^5$ theorem $\Rightarrow H_{\text{odd}} = 0$.
- We can replace the assumption using further structural results for torus representations with connected isotropy groups (c.i.g.):

**Theorem:** There exists $d < \infty$ such that any closed, orientable $M$ with positive curvature and a c.i.g. $T^d$-action is a rational $S$, $\mathbb{CP}$, $\mathbb{HP}$.

Question 2: Can we prove these results for $\mathbb{Z}_2$-cohomology?

- Need to improve the rational four-periodicity theorem to a $\mathbb{Z}_2$ analogue:

**Conjecture:** If $H^*(M^n; \mathbb{Z}_2)$ is $k$-periodic, then it is four-periodic.

- (K. '13) implies $H^*(M^n; \mathbb{Z}_2)$ is $2^a$-periodic, generalizing (Adem '52).
- If $k$ divides $n$, (Adams '60) implies four-periodic.
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