Torus actions and positive curvature

Lee Kennard

Syracuse University

Joint with **Michael Wiemeler** (Münster) and **Burkhard Wilking** (Münster)

Virtual seminar on geometry with symmetries

6 May 2020

Positive curvature and symmetry

Setup: (M, g) – closed, orientable, $\sec > 0$ (e.g., $\mathbb{S}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, $\mathbb{OP}^2$).

(Other known examples in $\dim M \in \{6, 7, 12, 13, 24\}$)

Conjecture (Hopf 1930s): $\sec > 0 \Longrightarrow M \neq \mathbb{S}^2 \times \mathbb{S}^2$.

Conjecture (Hopf 1930s): $\sec > 0 \Longrightarrow \chi(M^{2n}) > 0$.

Grove symmetry program (1990s): Study $\sec > 0$ with symmetry.

(Homogeneous spaces, cohomogeneity one manifolds, quotients,. . . .)

Major developments:

- (Shankar '98) Resolution of Chern's 1965 question on $\pi_1(M)$.
- (Dearricott '11, Grove-Verdiani-Ziller '11) New example with $\sec > 0$.
- (Gromoll-Meyer '74, Grove-Ziller '00, Goette-Kerin-Shankar '20)

  Many 2-connected 7-manifolds admit $\sec \geq 0$, including all exotic $\mathbb{S}^7$.

Positive curvature and torus symmetry

Setup: (M^{2n}, g) – closed, orientable, $\sec > 0$ (e.g., $\mathbb{S}^{2n}$, $\mathbb{CP}^n$, $\mathbb{HP}^{\frac{n}{2}}$, $\mathbb{OP}^2$).

Conjecture (Hopf 1930s): $\sec > 0 \implies \chi(M^{2n}) > 0$.

Today's symmetry assumption: T^d acts isometrically with $d \geq C(n)$.

Theorem (Grove-Searle '94): $C(n) = n \implies$ diffeomorphism rigidity.

Theorem (Wilking '03): $C(n) = \frac{n}{2} + 1 \implies$ homotopy rigidity.

Theorem (Amann-K. '14): $C(n) = 2\log_2(2n)$, $b_3 = 0 \Rightarrow \chi(M) > 0$.

Theorem (K.-Wiemeler-Wilking): $C(n) = \log_2(2n) \Rightarrow \chi(M)$ standard.

Theorem (K.-Wiemeler-Wilking): $C(n) = 5 \implies \chi(M) > 0$.

Compare: (Rong '99), (Dessai '05 & '07), (Weisskopf '17).

Tools, I: Existing tools and eliminating "$b_3 = 0$"

Setup: (M^{2n}, g) – closed, orientable, $\sec > 0$, invariant under T^d.

General structure theory

(General theory of transformation groups + classical results from $\sec > 0$.)

+ **Connectedness & periodicity lemmas (Wilking '03)**

(Morse theory of geodesics + second variation of energy)

+ **Rational four-periodicity theorem (K. '13)**

(Steenrod squares and Steenrod powers; implies 4-periodic Betti numbers)

+ **b_3 lemma (K.-Wiemeler-Wilking)**

(Equivariant cohomology & global analysis of fixed point set)

$\Downarrow$ Bake at 350°F

For $F \subseteq M^{\mathsf{T}^d}$, it suffices* to find $\mathsf{T}^d \supseteq \mathsf{H}_i \supseteq \mathsf{H}$ with $\mathsf{H}_i/\mathsf{H} \cong \mathsf{S}^1$ such that the fixed point components $F \subseteq N_i \subseteq N$ satisfy $N_1 \pitchfork N_2$.

Tools, II: Reducing $C(n)$ to 5

Question: How do we get transverse intersections?

Fix $p \in F \subseteq M^{\mathrm{T}^d}$, and study isotropy representation $\rho : \mathrm{T}^d \to \mathrm{SO}(V)$.

We need to find $\mathrm{T}^d \supseteq \mathrm{H}_i \supseteq \mathrm{H}$ such that $V^{\mathrm{H}_1} \pitchfork V^{\mathrm{H}_2}$ in V^{H}.

(If $C(n) \approx \log_2 n$, one can look at subgroups of $\mathbb{Z}_2^d \subseteq \mathrm{T}^d$.)

S^1**-splitting (K.-Wiemeler-Wilking):** If $\rho : \mathrm{T}^d \to \mathrm{SO}(V)$ is faithful and $d \geq 3$, there exists $\mathrm{H} \subseteq \mathrm{T}^d$ such that the induced representation $\mathrm{T}^{d-1} = \mathrm{T}^d/\mathrm{H} \to \mathrm{SO}(V^{\mathrm{H}})$ factors through a product representation $\mathrm{S}^1 \times \mathrm{T}^{d-2} \to \mathrm{SO}(V_1) \times \mathrm{SO}(V_2)$ for some decomposition $V^{\mathrm{H}} = V_1 \oplus V_2$.

In V^{H}, the fixed point sets of S^1 and T^{d-2} intersect transversely.

* If $d \geq 5$, one can iterate to gain control over dimensions of $V^{\mathrm{H}_i} \subseteq V^{\mathrm{H}}$.

S^1 splitting, I: Examples & Reduction

S^1-**splitting**: Given $\rho : \mathrm{T}^d \to \mathrm{SO}(V)$ with $d \geq 3$, there exists $\mathrm{H} \subseteq \mathrm{T}^d$ such that the induced representation $\mathrm{T}^{d-1} = \mathrm{T}^d/\mathrm{H} \to \mathrm{SO}(V^{\mathrm{H}})$ factors through $\mathrm{S}^1 \times \mathrm{T}^{d-2} \to \mathrm{SO}(V_1) \times \mathrm{SO}(V_2)$, for some $V^{\mathrm{H}} = V_1 \oplus V_2$.

Example 1: $\rho(z_1, z_2, z_3) = \mathrm{diag}(z_1, z_2, z_3, z_1\bar{z}_2, z_2\bar{z}_3, z_1\bar{z}_3) \in \mathrm{U}(6)$.

Take $\mathrm{H} = \{(z, z, 1)\} \quad \rightsquigarrow \quad V^{\mathrm{H}} = \mathbb{C}e_3 \oplus \mathbb{C}e_4$ and $\bar{\rho}$ splits.

Example 2: $\rho(a, b, c) = \mathrm{diag}(a, b, c, \underline{ab}, ac, bc, \underline{a\bar{b}}, a\bar{c}, b\bar{c}, ab\bar{c}, a\bar{b}c, \bar{a}bc)$.

First look at $\mathbb{Z}_2 \cong \langle(-1, -1, -1)\rangle \quad \rightsquigarrow \quad V^{\mathbb{Z}_2} = \mathbb{C}e_4 \oplus \ldots \oplus \mathbb{C}e_9$.

Then look at $\mathrm{H} = \mathbb{Z}_2 \cdot \{(1, 1, z)\}. \quad \rightsquigarrow \quad V^{\mathrm{H}} = \underline{\mathbb{C}e_4} \oplus \underline{\mathbb{C}e_7}. \quad \rightsquigarrow \quad \bar{\rho}$ splits

Observation: The splitting holds if there exists a finite isotropy group F.

(Look at induced representation $\mathrm{T}^d \cong \mathrm{T}^d/\mathrm{F} \to \mathrm{SO}(V^{\mathrm{F}})$. Use induction.)

Reduction: In the proof, we may assume connected isotropy groups (c.i.g.)

S^1-splitting, II: Torus representations with c.i.g.

Setup: Assume $\rho : T^d \to SO(V)$ has connected isotropy groups.

Non-example: The representation $\rho : T^3 \to U(6)$ given by

$$\rho(z_1, z_2, z_3) = \mathrm{diag}(z_1, z_1^3, z_2\bar{z}_3, \underline{z_2 z_3, z_1 z_3, z_1 z_2})$$

has disconnected isotropy groups

$$T^3_{e_2} \cong \mathbb{Z}_3 \times T^2 \ , \quad T^3_{e_3+e_4} \cong S^1 \times \mathbb{Z}_2 \ , \quad T^3_{\underline{e_4+e_5+e_6}} \cong \mathbb{Z}_2.$$

Example 1: $\rho : T^4 \to U(10) \subseteq SO(20)$ given by

$$\mathrm{diag}(z_1, z_2, z_3, z_4, z_1\bar{z}_2, z_1\bar{z}_3, z_1\bar{z}_4, z_2\bar{z}_3, z_2\bar{z}_4, z_3\bar{z}_4).$$

Example 2: $\rho : T^4 \to U(9) \subseteq SO(18)$ given by

$$\mathrm{diag}(z_1, z_2, z_3, z_4, z_1\bar{z}_3, z_1\bar{z}_4, z_2\bar{z}_3, z_2\bar{z}_4, z_1 z_2\bar{z}_3\bar{z}_4).$$

Classification for $d = 4$: Any torus representation with c.i.g. is equivalent to a subrepresentation of one of these examples (ignoring multiplicities).

S^1-splitting, III: Proof

Setup: Assume $\rho : T^d \hookrightarrow SO(V)$ has connected isotropy groups.

Initial analysis:

Build a matrix H whose columns are weights.

Lemma: c.i.g. $\iff$ every $d \times d$ submatrix H' has $\det(H') = 0$ or ± 1.

For example: $\rho(z,w) = (z, w, zw, \bar{z}w)$ has $H = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

Submatrix H' has det $= 2$, corresponding to

$$T^2_{e_3+e_4} = \ker(zw) \cap \ker(\bar{z}w) = \langle(-1,-1)\rangle \cong \mathbb{Z}_2.$$

We may assume the e_i are weights, so *every submatrix has* $\det = 0, \pm 1$.

In particular, the weights $h_i \in \{-1, 0, 1\}^d$ and $\# \{h_i\} \leq 3^d$.

S^1-splitting, III: Proof (cont.)

Further combinatorial analysis:

(1) # weights $\leq \frac{d(d+1)}{2}$, and this is sharp. Independent of dim V!

(2) There is a basis $h_1, \ldots, h_d$ of weights such that h_1 is the one and only weight in the affine subspace $h_1 + \langle h_3, \ldots, h_d \rangle$. (Implies the S^1-**splitting**.)

Example 1: $(z_1, z_2, z_3, z_4, z_1\bar{z}_2, z_1\bar{z}_3, z_1\bar{z}_4, z_2\bar{z}_3, z_2\bar{z}_4, z_3\bar{z}_4)$ has 10 weights.

e_1 is the only weight in $e_1 + \langle e_2 - e_3, e_3 - e_4 \rangle$, so $H = \{(1, z, z, z)\}$ gives rise to a splitting $\bar{\rho} : T^3 \to \mathrm{U}(4)$ of the form $\bar{\rho}(x, y, z) = \mathrm{diag}(x, y, z, yz)$.

Example 2: $(z_1, z_2, z_3, z_4, z_1\bar{z}_3, z_1\bar{z}_4, z_2\bar{z}_3, z_2\bar{z}_4, z_1 z_2\bar{z}_3\bar{z}_4)$.

Only one weight in $e_1 + e_2 - e_3 - e_4 + \langle e_2, e_3 \rangle$, so take $H = \{(\bar{z}, 1, 1, z)\}$.

Proof of main theorem

Setup: (M^{2n}, g) is closed, oriented, positively curved with T^5 symmetry.

Apply S^1**-splitting** twice to the isotropy representation at $p \in F \subseteq M^{\mathsf{T}^5}$:

There exist $\mathsf{H}^2 \subseteq \mathsf{T}^5$ and three circles $\mathsf{S}^1_i \subseteq \mathsf{T}^5/\mathsf{H}$ such that $N_p^{\mathsf{S}^1_i} \pitchfork N_p^{\mathsf{S}^1_j}$.

4-periodicity + b_3 **lemma** $\Rightarrow N = M_p^{\mathsf{H}} \sim_{\mathbb{Q}} \mathbb{S}^m, \mathbb{CP}^m, \mathbb{HP}^m, \mathbb{S}^2 \times \mathbb{HP}^m$.

Localization theorem $\Rightarrow F \sim_{\mathbb{Q}} \mathbb{S}^l, \mathbb{CP}^l, \mathbb{HP}^l, \mathbb{S}^2 \times \mathbb{HP}^l, \mathbb{S}^2 \times \mathbb{CP}^l$.

- $F \not\sim_{\mathbb{Q}} \mathbb{S}^2 \times \mathbb{CP}^l$ is easy (cohomology is not periodic).
- $F \not\sim_{\mathbb{Q}} \mathbb{S}^2 \times \mathbb{HP}^l$ is hard (global analysis of M^{T^5} and isotropy weights).

Related results (K.-Wiemeler-Wilking)

Theorem 1: If T^5 acts on a closed, orientable, positively curved M^{2n}, then every fixed point component of T^5 is a rational $\mathbb{S}^k$, $\mathbb{CP}^k$, or $\mathbb{HP}^k$.

Why work so hard for the cohomology of M^{T^5}?

Theorem 2: If M^n (closed, orientable) admits an *equivariantly formal* T^8-action such that every fixed point component of every $\mathsf{T}^5 \subseteq \mathsf{T}^8$ is a rational $\mathbb{S}$, $\mathbb{CP}$, or $\mathbb{HP}$, then M is a rational $\mathbb{S}^n$, $\mathbb{CP}^{\frac{n}{2}}$, or $\mathbb{HP}^{\frac{n}{4}}$.

- Partial converse to (Smith '38, Bredon '64): Fixed point components of torus actions on $\mathbb{S}$ / $\mathbb{CP}$ / $\mathbb{HP}$ are again $\mathbb{S}$ / $\mathbb{CP}$ / $\mathbb{HP}$.
- Special case (GKM action): $\dim(M^{\mathsf{T}^8}) = 0$ and every $\dim(M^{\mathsf{T}^7}) \leq 2$. We use results from Goertsches-Wiemeler '15.

Corollary: If T^8 acts on a closed, orientable, positively curved M^{2n} with $H^{\mathrm{odd}}(M;\mathbb{Q}) = 0$, then M is a $\mathbb{Q}$-cohomology $\mathbb{S}^{2n}$, $\mathbb{CP}^n$, or $\mathbb{HP}^{\frac{n}{2}}$.

Ongoing work

Question 1: Do we need the assumption $H^{\text{odd}}(M;\mathbb{Q}) = 0$ in the T^8 result?

- The Bott-Grove-Halperin conjecture + the T^5 theorem $\Rightarrow H^{\text{odd}} = 0$.
- We can replace the assumption using further structural results for torus representations with connected isotropy groups (c.i.g.):

Theorem: There exists $d < \infty$ such that any closed, orientable M with positive curvature and a c.i.g. T^d-action is a rational $\mathbb{S}$, $\mathbb{CP}$, $\mathbb{HP}$.

Question 2: Can we prove these results for $\mathbb{Z}_2$-cohomology?

- Need to improve the rational four-periodicity theorem to a $\mathbb{Z}_2$ analogue:

Conjecture: If $H^*(M^n;\mathbb{Z}_2)$ is k-periodic, then it is four-periodic.

- (K. '13) implies $H^*(M^n;\mathbb{Z}_2)$ is 2^a-periodic, generalizing (Adem '52).
- If k divides n, (Adams '60) implies four-periodic.

References

M. Amann and L. Kennard.
Topological properties of positively curved manifolds with symmetry.
Geom. Funct. Anal., 24(5):1377–1405, 2014.

G.E. Bredon.
The cohomology ring structure of a fixed point set.
Ann. of Math. (2), 80:524–537, 1964.

O. Dearricott.
A 7-manifold with positive curvature.
Duke Math. J., 158(2):307–346, 2011.

A. Dessai.
Characteristic numbers of positively curved spin-manifolds with symmetry.
Proc. Amer. Math. Soc., 133(12):3657–3661, 2005.

A. Dessai.
Obstructions to positive curvature and symmetry.
Adv. Math., 210(2):560–577, 2007.

S. Goette, M. Kerin, and K. Shankar.
Highly connected 7-manifolds and non-negative sectional curvature.
Ann. of Math. (2), 191(3):829–892, 2020.

D. Gromoll and W. Meyer.
An exotic sphere with nonnegative sectional curvature.
Ann. of Math. (2), 100:401–406, 1974.

K. Grove and C. Searle.
Positively curved manifolds with maximal symmetry rank.
J. Pure Appl. Algebra, 91(1):137–142, 1994.

K. Grove, L. Verdiani, and W. Ziller.
An exotic $T_1\mathbb{S}^4$ with positive curvature.
Geom. Funct. Anal., 21(3):499–524, 2011.

O. Goertsches and M. Wiemeler.
Positively curved GKM-manifolds.
Int. Math. Res. Not. IMRN, (22):12015–12041, 2015.

K. Grove and W. Ziller.
Curvature and symmetry of Milnor spheres.
Ann. of Math. (2), 152(1):331–367, 2000.

W.-Y. Hsiang and J.C. Su.
On the geometric weight system of topological actions on cohomology quaternionic projective spaces.
Invent. Math., 28:107–127, 1975.

L. Kennard.
On the Hopf conjecture with symmetry.
Geom. Topol., 17:563–593, 2013.

X. Rong.
Positive curvature, local and global symmetry, and fundamental groups.
Amer. J. Math., 121:931–943, 1999.

K. Shankar.
On the fundamental groups of positively curved manifolds.
J. Differential Geom., 49(1):179–182, 1998.

P.A. Smith.
Transformations of a finite period.
Ann. of Math., 39:127–164, 1938.

N. Weisskopf.
Positive curvature and the elliptic genus.
New York J. Math., 23:193–212, 2017.

B. Wilking.
Torus actions on manifolds of positive sectional curvature.
Acta Math., 191(2):259–297, 2003.