OBSTRUCTIONS TO FREE ACTIONS ON BAZAIKIN SPACES

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Abstract. Apart from spheres and an infinite family of manifolds in dimension seven, Bazaikin spaces are the only known examples of simply connected Riemannian manifolds with positive sectional curvature in odd dimensions. We consider positively curved Riemannian manifolds whose universal covers have the same cohomology as Bazaikin spaces and prove structural results for the fundamental group in the presence of torus symmetry.

1. Introduction

An important question in Riemannian geometry is to investigate the structure of fundamental groups of Riemannian manifolds with non-negative sectional curvature. A well-known example of this is a theorem of Gromov which states that the fundamental group of a complete Riemannian manifold M^n with non-negative sectional curvature has at most $C(n)$ generators, where $C(n)$ is a constant depending only on the dimension of M (see [Gro78]). In addition, the Cheeger–Gromoll splitting theorem, together with a theorem of Wilking, implies that a group G is the fundamental group of a non-negatively curved Riemannian manifold if and only if G has a normal subgroup isomorphic to $\mathbb{Z}^d$ such that the quotient group is finite (see [CG71] and [Wil00, Thm. 2.1]).

Under the stronger assumption of positive curvature, the only known further obstructions are the results of Bonnet–Myers and Synge which together imply that the fundamental group of a positively curved Riemannian manifold is finite and, moreover, trivial or $\mathbb{Z}_2$ if the dimension of the manifold is even.

As for examples, the largest class of groups which arise as fundamental groups of positively curved manifolds are the spherical space form groups. These are groups that act freely and linearly on spheres (for a complete classification, see [Wol11, Chap. III]). The first step in the classification of spherical space form groups is to establish that they satisfy the (p^2) and $(2p)$ conditions, which mean respectively that every subgroup of order p^2 or $2p$ is cyclic. The (p^2) condition was proved by Smith for groups acting freely on a mod p homology sphere, i.e., a space whose homology groups with coefficients in $\mathbb{Z}_p$ coincide with that of a sphere (see [Smi44]). Moreover, the $(2p)$ condition holds for groups acting freely on a mod 2 homology

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sphere by results of Milnor and Davis (see [Mil57] and [Dav83]).

In 1965, Chern asked if the (p^2) condition holds for the fundamental groups of Riemannian manifolds with positive sectional curvature. This question was not answered for over 30 years until Shankar proved that there are examples for which the (p^2) condition fails for $p = 2$ and the $(2p)$ condition fails for all p (see [Sha98]). Later, Bazaikin and Grove–Shankar (see [Baz99] and [GS00]) showed that there are other classes of positively curved manifolds which fail to satisfy the (p^2) condition for $p = 3$. It remains an open problem whether the (p^2) condition holds for $p \geq 5$ (see [GSZ06]).

In the presence of symmetry, much more is known. For example, the only groups that can arise as fundamental groups of Riemannian homogeneous spaces with positive sectional curvature are finite subgroups of SO(3) or SU(2) (see [WZ18]). Under more relaxed symmetry assumptions, the most remarkable result in this direction is due to Rong (see [Ron99]): The fundamental group of an odd-dimensional positively curved Riemannian manifold M with circle symmetry has a cyclic subgroup of index at most a constant $w(n)$ depending only on the dimension of M. In particular, the (p^2) condition holds for sufficiently large p for this class of manifolds. The constant $w(n)$ here is larger than Gromov's Betti number estimate. Part of our motivation is to refine the estimate for $w(n)$ in dimension 13. Our main result replaces S^1 by T^2 or T^3 and restricts to the class of manifolds whose universal covers have the rational cohomology of a Bazaikin space (see [Baz96] and [FZ09]).

Theorem A. *Let M^{13} be a closed Riemannian manifold with positive sectional curvature. Suppose that the universal cover of M is a rational cohomology Bazaikin space.*

(1) *If M admits an effective isometric T^2-action, then $\pi_1(M)$ has a cyclic subgroup whose index D either is 27 or divides 18. Moreover, $D \leq 9$ if the universal cover of M is also a mod 3 cohomology Bazaikin space.*
(2) *If M admits an effective isometric T^3-action, then $\pi_1(M)$ has a cyclic subgroup of index at most three.*

Remark.

- Note that $\pi_1(M)$ is cyclic in Theorem A under the stronger assumption of T^4 symmetry by a result of Frank, Rong, and Wang (see [FRW13]).
- Since the index of cyclic subgroup in Theorem A is not divisible by primes greater than three, it follows that under the assumptions of Theorem A, $\pi_1(M)$ satisfies the (p^2) condition for all $p \geq 5$. Davis has commented to the author that the results in [DM91] imply that, for all odd primes p, there exists a closed, simply connected, smooth manifold with the rational cohomology of a Bazaikin space that admits a free action by $\mathbb{Z}_p \times \mathbb{Z}_p$. Therefore Theorem A does not hold without the curvature and symmetry assumptions.

We now discuss a corollary to Theorem A. The only simply connected, closed 13-dimensional manifolds known to admit positive curvature are $\mathbb{S}^{13}$ and the Bazaikin spaces (see [Zil07]). By a result of Kennard (see [Ken17, Cor. 6.3]), if M^{13} is a closed, positively curved Riemannian manifold with T^2 symmetry whose universal cover is a rational sphere, then $\pi_1(M)$ is cyclic. Combining this result with Theorem A, we get the following corollary:

Corollary. *If M^{13} is a closed, positively curved Riemannian manifold with T^2 symmetry, and if the universal cover of M has the integral cohomology of one of the known positively curved 13-dimensional examples, then $\pi_1(M)$ has a cyclic subgroup of index $1, 2, 3, 6$, or 9.*

We now discuss the tools used in the proof. In addition to the well known results such as Berger's fixed point theorem and Wilking's connectedness lemma (see Section 2), the main new tool is Lemma 4.2. This is a structural result for groups acting freely on positively curved manifolds with circle symmetry that generalizes an obstruction from Kennard (see [Ken17, Prop. 5.1]). Together with a result of Davis and Weinberger (see Theorem 2.7), Lemma 4.2 places strong restrictions on the Sylow subgroups of the fundamental group. In fact, we show that after possibly passing to a subgroup B of index $2, 3, 6$, or 9, every Sylow subgroup is cyclic. Burnside's classification (see Section 2), together with Lemma 4.2, then implies that B itself has a cyclic subgroup of index 1 or 3. In addition, results from equivariant cohomology are applied to calculate the fixed point components of the circle action. The key here is that we fix the rational type of the universal cover. Finally, to analyze the case in which the manifold is a mod 3 cohomology Bazaikin space, we modify an argument due to Heller to further restrict the Sylow 3-subgroups of the fundamental group (see Section 6).

This article is organized as follows. Section 2 provides basic results which will be used throughout the paper. Section 3 states the definition of Bazaikin spaces as well as a lemma about groups acting freely and isometrically on a rational cohomology Bazaikin space. In Section 4, we prove Lemma 4.2. Theorem A in the case of rational cohomology Bazaikin spaces is proved in Section 5. Finally, in Section 6, we complete the proof of Theorem A by considering the case of mod 3 cohomology Bazaikin spaces.

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2. Preliminaries

This section consists of three parts. The first part states some results about positively curved manifolds. In the second part, we provide a theorem from equivariant cohomology. The last part discusses some tools from group theory.

One of the most powerful results in the theory of positively curved manifolds is the following theorem due to Wilking:

Theorem 2.1 (Connectedness lemma, [Wil03, Thm. 2.1])**.** *Let M^n be a closed positively curved Riemannian manifold. If N^{n-k} is a closed totally geodesic submanifold of M, then the inclusion $N^{n-k} \hookrightarrow M^n$ is $(n-2k+1)$-connected. Moreover, if N^{n-k} is fixed pointwise by the action of a Lie group G which acts isometrically on M, then the inclusion is $(n-2k+1+\delta(G))$-connected, where $\delta(G)$ is the dimension of the principal orbit.*

The following is a result of Poincaré duality and refines the conclusion of Theorem 2.1:

Lemma 2.2 ([Wil03, Lem. 2.2])**.** *Let M^n be a closed orientable smooth manifold and let N^{n-k} be a closed orientable submanifold. If the inclusion $N^{n-k} \hookrightarrow M^n$ is $(n-k-l)$-connected and $n-k-2l > 0$, then there exists $e \in H^k(M;\mathbb{Z})$ such that the map $\cup e : H^i(M;\mathbb{Z}) \to H^{i+k}(M;\mathbb{Z})$ is surjective for $l \leq i < n-k-l$ and injective for $l < i \leq n-k-l$.*

The next result is due to Frank, Rong, and Wang.

Proposition 2.3 ([FRW13, Cors. 1.7 and 1.9])**.** *Suppose that M^n is a closed odd-dimensional Riemannian manifold with positive sectional curvature. If $n \geq 5$ and M has a closed totally geodesic submanifold N of codimension two, then the universal covering spaces of M and N are homotopy spheres, and $\pi_1(M) \cong \pi_1(N)$ is cyclic.*

For our purposes, we need the following generalization of Proposition 2.3:

Proposition 2.4. *Let M^n be a closed odd-dimensional Riemannian manifold with positive sectional curvature. Suppose that $n \geq 5$ and M has a closed totally geodesic submanifold N of codimension two. If Γ is a finite group that acts freely and isometrically on M such that the action preserves N, then Γ is cyclic.*

Proof. Consider the Riemannian covering map $q : M \to M/\Gamma$. By [Hat02, Prop. 1.40], we have $\Gamma \cong \pi_1(M/\Gamma)/q_*(\pi_1(M))$. In addition, M/Γ and N/Γ are closed odd-dimensional positively curved manifolds and N/Γ is a totally geodesic submanifold of M/Γ of codimension two. Hence $\pi_1(M/\Gamma)$ is cyclic by Proposition 2.3. This implies that $\Gamma \cong \pi_1(M/\Gamma)/q_*(\pi_1(M))$ is cyclic. □

We end the first part of this section with a generalization of Berger's theorem about torus actions on positively curved Riemannian manifolds of even dimension (see [Ber61]). The statement in odd dimensions is due to Sugahara.

Theorem 2.5 ([Sug82], cf. [GS94])**.** *Let M be a closed odd-dimensional Riemannian manifold with positive sectional curvature. If M admits an effective isometric T^k-action, then there is a circle orbit. In particular, there exists $T^{k-1} \subseteq T^k$ with non-empty fixed point set.*

One of the main tools used in the proof of Theorem A is the relationship between the cohomology of a manifold M and that of the fixed point set M^{S^1} of a circle acting on M. For our purposes, we need the following result. It is proved by applying tools from equivariant cohomology.

Theorem 2.6 ([AP93, Thms. 3.8.12 and 3.10.4])**.** *If M is a compact manifold, which admits a smooth S^1-action, then the rational Betti numbers satisfy*

$$\sum_i b_i(M^{S^1};\mathbb{Q}) \leq \sum_i b_i(M;\mathbb{Q}).$$

Moreover, if $\sum_i b_i(M^{S^1};\mathbb{Q}) = \sum_i b_i(M;\mathbb{Q})$ and if $H^*(M;\mathbb{Q})$ has r generators of even degree and s generators of odd degree, then for any component F of M^{S^1}, $H^*(F;\mathbb{Q})$ has at most r generators of even degree and at most s generators of odd degree.

We end this section with some results from the theory of finite groups and free actions by finite groups. The first result is due to Davis and Weinberger.

Theorem 2.7 ([Dav83, Thm. D])**.** *Let M^{4k+1} be a closed manifold such that the integer $\sum_{i=0}^{2k}(-1)^i \dim H^i(M;\mathbb{Q})$ is odd. If G is a finite group that acts freely on M such that the induced action on $H^*(M;\mathbb{Q})$ is trivial, then G is the direct product of a cyclic 2-group and a group Γ of odd order.*

Remark 2.8. For G and Γ as in Theorem 2.7, G is cyclic if and only if Γ is cyclic, and, more generally, G has a cyclic subgroup of index r if and only if Γ has a cyclic subgroup of index r.

In the proof of Theorem A, we are interested in the Sylow p-subgroups of $\pi_1(M)$. Theorem 2.7 states a condition under which the Sylow 2-subgroup of a finite group acting freely on a $(4k+1)$-dimensional manifold is cyclic. Theorem 2.10 provides a condition under which Sylow p-subgroups for odd p are cyclic. Before proceeding, we recall the definition of the (p^2) and $(2p)$ conditions.

Definition 2.9. Let Γ be a finite group and let p be a prime. We say that Γ satisfies

- (p^2) condition if every subgroup of order p^2 is cyclic.
- $(2p)$ condition if every subgroup of order $2p$ is cyclic.

Theorem 2.10 ([Wol11, Thm. 5.3.2])**.** *If Γ is a finite group of odd order, then the following statements are equivalent:*

(1) *Γ satisfies every (p^2) condition.*
(2) *Every Sylow p-subgroup of Γ is cyclic.*

Odd-order groups which satisfy all (p^2) conditions have a nice presentation and enjoy some properties which will be discussed in what follows:

Theorem 2.11 (Burnside, [Wol11, Thm. 5.4.1])**.** *If G is a finite group in which every Sylow subgroup is cyclic, then G is generated by two elements A and B with defining relations*

$$A^m = B^n = 1, \quad BAB^{-1} = A^r;$$
$$((r-1)n, m) = 1, \quad r^n \equiv 1 (\mathrm{mod}\ m).$$

Definition 2.12. The collection of all groups of the form

$\langle A, B : A^m = B^n = 1, BAB^{-1} = A^r \rangle$ where $((r-1)n, m) = 1$ and $r^n \equiv 1(\mathrm{mod}\ m)$

will be denoted by $\mathcal{C}$. We partition the collection $\mathcal{C}$ into groups $\mathcal{C}_d$, where d denotes the order of r in the multiplicative group of units modulo m.

Remark 2.13. Note that every $\Gamma \in \mathcal{C}_d$ has a normal cyclic subgroup of index d. Indeed, the subgroup H generated by A and B^d is a normal cyclic subgroup of index d in Γ (see [Wol11, Thm. 5.5.1]). It can be proved moreover that H is not strictly contained in any cyclic subgroup.

The last collection of algebraic tools which we require are some basic results about p-groups and normal p-complements. Let p be a prime. It is a well-known fact that every p-group P with $|P| = p^m$ has a normal subgroup of order p^i for all $1 \le i \le m$. Moreover, the classification of groups of order p^3 (see [Bur55, p. 140]) implies that any group of order 27 is isomorphic to $\mathbb{Z}_{27}$, $\mathbb{Z}_9 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$, or

$$U(3,3) := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z}_3 \right\}.$$

We also require a result about groups of order 81. Every non-cyclic group of order 81 contains either a copy of $\mathbb{Z}_9 \times \mathbb{Z}_3$ or a copy of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ (see [Bur55, pp. 140 and 145]). These facts imply the following proposition:

Proposition 2.14. *If G is a 3-group which contains $\mathbb{Z}_3 \times \mathbb{Z}_3$ but does not contain $\mathbb{Z}_9 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, or $U(3,3)$, then G is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$.*

Proof. By the discussion above, we only need to prove that the order of G is at most 81. Suppose by way of contradiction that the order of G is bigger than 81 and let P_{81} be a normal subgroup of G of order 81. If P_{81} is non-cyclic, then it contains a copy of $\mathbb{Z}_9 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, a contradiction. Hence we may assume that $P_{81} = \langle g \rangle$ is cyclic. Since P_{81} is cyclic and G contains a copy of $\mathbb{Z}_3 \times \mathbb{Z}_3$, there exists $\mathbb{Z}_3 = \langle h \rangle \subseteq G$ such that $\langle h \rangle \cap P_{81} = \{1\}$. Since $\langle h \rangle$ normalizes $\langle g^3 \rangle$, we can form the subgroup $K = \langle g^3 \rangle \langle h \rangle$ which is a group of order 81. By the information concerning groups of order 81 mentioned above, in order to get a contradiction, it suffices to show that K is non-cyclic.

Suppose for a moment that K is cyclic and hence abelian. Then every element in K is of the form $(g^3)^i h^j$ and has order at most 27, a contradiction. $\square$

The normal rank of a p-group P is the largest integer k such that P contains an elementary abelian normal subgroup of order p^k. Our final algebraic result is the following:

Theorem 2.15 ([Gor80, p. 257])**.** *Let G be a finite group and let p be the smallest prime dividing the order of G. Let P denote the Sylow p-subgroup of G. Suppose that P is cyclic if $p = 2$ and that the normal rank of P is at most two otherwise. Then there exists a normal p-complement of G, i.e., a normal subgroup N of G such that $G = PN$ and $P \cap N = \{1\}$.*

3. Bazaikin Spaces

Besides spherical space forms, the only 13-dimensional manifolds known to admit positive curvature are a family of biquotients called Bazaikin spaces.

Biquotients are defined in the following way. Let G be a compact Lie group and let U be a subgroup of $G \times G$. There exists an action of U on G defined by $(u_1, u_2) \cdot g = u_1 g u_2^{-1}$. In case the action is free, the quotient space is called a biquotient and is denoted by $G /\!/ U$.

Bazaikin spaces are examples of biquotients but here we give a slightly different description (for more details, see [FZ09]). Let $q = (q_1, \ldots, q_5)$ be a five tuple of integers and let $q_0 := \sum q_i$. There exists an injective homomorphism

$$\begin{aligned}&\mathrm{Sp}(2) \times S^1 \to \mathrm{U}(5) \times \mathrm{U}(5),\\ &\qquad (A, z) \mapsto (\mathrm{diag}(z^{q_1}, \dots z^{q_5}), \mathrm{diag}(z^{q_0}, A)),\end{aligned}$$

where we consider Sp(2) as a subgroup of SU(4) via the inclusion

$$A + Bj \in \mathrm{Sp}(2) \longmapsto \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}.$$

The above homomorphism gives an action of $\mathrm{Sp}(2) \times S^1$ on U(5) defined by $(A, z) \cdot g = \mathrm{diag}(z^{q_1}, \dots z^{q_5}) g \,\mathrm{diag}(z^{-q_0}, A^{-1})$ which restricts to an action of $\mathrm{Sp}(2) \times S^1$ on SU(5). The kernel of this action is $\mathbb{Z}_2$ and hence we obtain an effective action of $\mathrm{Sp}(2) \cdot S^1 := (\mathrm{Sp}(2) \times S^1)/\mathbb{Z}_2$ on SU(5). The action of $\mathrm{Sp}(2) \cdot S^1$ on SU(5) is free if and only if all the q_i are odd and $\gcd(q_{\sigma(1)} + q_{\sigma(2)}, q_{\sigma(3)} + q_{\sigma(4)}) = 2$ for all permutations $\sigma \in S_5$. In this case, the quotient space $B_q = \mathrm{SU}(5)/\mathrm{Sp}(2) \cdot S^1$ is called a Bazaikin space. The Bazaikin space B_q admits positive sectional curvature if $q_i + q_j > 0$ (or < 0) for all $i < j$.

Proposition 3.1 ([Baz96])**.** *The integral cohomology groups of the Bazaikin space B_q are given by:*

$$H^k(B_q; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2, 4, 9, 11, 13, \\ \mathbb{Z}_m & \text{if } k = 6, 8, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $m = \frac{1}{8} \sum_{i<j<k} q_i q_j q_k$. Moreover, if x is a generator of $H^2(B_q; \mathbb{Z})$, then x^k is a generator of $H^{2k}(B_q; \mathbb{Z})$ for $k = 2, 3, 4$. In particular, B_q has the rational cohomology of $\mathbb{C}\mathrm{P}^2 \times \mathbb{S}^9$ and the mod 3 cohomology of either $\mathbb{C}\mathrm{P}^2 \times \mathbb{S}^9$ or $\mathbb{C}\mathrm{P}^4 \times \mathbb{S}^5$.

One of the crucial steps in the proof of Theorem A, is to find a subgroup of $\pi_1(M)$ of minimal index which satisfies the conditions of Theorem 2.7. The following lemma provides such a subgroup:

Lemma 3.2. *Let M^{13} be a closed Riemannian manifold with positive sectional curvature whose universal cover has the rational cohomology of a Bazaikin space. Let $\Gamma := \pi_1(M)$. Then Γ has a subgroup Γ_1 of index at most two that acts trivially on $H^*(\widetilde{M}; \mathbb{Q})$, where $\widetilde{M}$ denotes the universal cover of M.*

Proof. Since Γ acts isometrically and freely on $\widetilde{M}$, Weinstein's theorem (see [dC92, Chap. 9, Thm. 3.7]) implies that Γ acts by orientation-preserving homeomorphisms on $\widetilde{M}$ and hence trivially on $H^{13}(\widetilde{M}; \mathbb{Q})$. Now, $H^*(\widetilde{M}; \mathbb{Q}) \cong H^*(\mathbb{C}\mathrm{P}^2 \times \mathbb{S}^9; \mathbb{Q})$ by Proposition 3.1, so $H^*(\widetilde{M}; \mathbb{Q})$ is generated by some $\alpha \in H^2(\widetilde{M}; \mathbb{Q})$ and $\beta \in H^9(\widetilde{M}; \mathbb{Q})$. Since Γ acts by homomorphisms on $H^*(\widetilde{M}; \mathbb{Q})$, there is a subgroup $\Gamma_1 \trianglelefteq \Gamma$ of index at most two fixing α. Indeed, Γ_1 is the kernel of the homomorphism $f : \Gamma \to \mathbb{Z}_2 = \{\pm 1\}$ defined by $f(\gamma) = \epsilon_\gamma$, where $\gamma^*(\alpha) = \epsilon_\gamma \alpha$. Since Γ_1 also fixes α^2 and $\alpha^2 \beta \in H^{13}(\widetilde{M}; \mathbb{Q})$, it acts trivially on $H^*(\widetilde{M}; \mathbb{Q})$. □

4. A new obstruction

One of the key tools in the proof of Theorem A has to do with the structure of finite groups which act freely and by isometries on a positively curved manifold with circle symmetry. Here, we generalize an obstruction from Kennard [Ken17, Prop. 5.1] (cf. Sun–Wang [SW09, Lem. 1.5]) to restrict the structure of such groups.

Definition 4.1. Let G be a group and let (X, A) be a pair of G-spaces. For $g \in G$, the Lefschetz number $\mathrm{Lef}(g; X, A)$ of g is defined as follows:

$$\mathrm{Lef}(g; X, A) = \sum_i (-1)^i \operatorname{tr}(g^* : H^i(X, A; \mathbb{Q}) \to H^i(X, A; \mathbb{Q})).$$

Lemma 4.2. *Let M be a closed positively curved Riemannian manifold. Let Γ be a group of odd order which acts freely and by isometries on M. Assume M admits an effective isometric S^1-action which commutes with the action of Γ. If $H = \langle\alpha\rangle$ is a normal cyclic subgroup of Γ that is not strictly contained in a larger cyclic subgroup, then $|\Gamma/H|$ divides the Lefschetz number $\mathrm{Lef}(\alpha; \bar{M}, M^{S^1})$ of α, where $\bar{M} = M/S^1$.*

Note that when applied to $\Gamma = \mathbb{Z}_p \times \mathbb{Z}_p$, Lemma 4.2 implies that p divides $\mathrm{Lef}(\alpha; \bar{M}, M^{S^1})$. Recall also from Remark 2.13 that for $\Gamma \in \mathcal{C}_d$, the subgroup H generated by A and B^d is a normal cyclic subgroup of index d in Γ that is not strictly contained in any cyclic subgroup. Hence Lemma 4.2 applies to $\Gamma \in \mathcal{C}_d$ and shows that d divides $\mathrm{Lef}(\alpha; \bar{M}, M^{S^1})$.

Proof. Since the action of α on M commutes with the circle action, α acts on $\bar{M}$. Consider the fixed point set $\bar{M}^\alpha$. We claim that $\chi(\bar{M}^\alpha) = \mathrm{Lef}(\alpha; \bar{M}, M^{S^1})$. In order to prove this, note that since the action of Γ on M commutes with the S^1-action, α acts on M^{S^1}. Moreover, the action of α on M^{S^1} is free since Γ acts freely on M. Therefore, $(M^{S^1})^\alpha = \varnothing$ and $\chi(\bar{M}^\alpha) = \chi(\bar{M}^\alpha, (M^{S^1})^\alpha)$, where $\chi(\bar{M}^\alpha, (M^{S^1})^\alpha) = \sum_i (-1)^i \dim H^i(\bar{M}^\alpha, (M^{S^1})^\alpha; \mathbb{Q})$. The claim follows since we have $\chi(\bar{M}^\alpha, (M^{S^1})^\alpha) = \mathrm{Lef}(\alpha; \bar{M}, M^{S^1})$ by [Hat02, Exercise 2.C.4] (cf. [AP93, p. 250]).

We may assume that $\bar{M}^\alpha$ is non-empty since otherwise $\mathrm{Lef}(\alpha; \bar{M}, M^{S^1})$ is zero by the claim and we are done. Since $H = \langle\alpha\rangle$ is a normal subgroup of Γ, Γ acts on $\bar{M}^\alpha$. The kernel of this action is H and hence it induces an action of $G = \Gamma/H$ on $\bar{M}^\alpha$. Note that G acts on the set of components of $\bar{M}^\alpha$ and hence partitions the set of components into orbits. Let $\{F_i\}_{i=1}^m$ be the set of connected components of $\bar{M}^\alpha$ and let G_{F_i} denote the isotropy group of F_i. Either $G_{F_i} = 1$ for all i or $G_{F_i} \neq 1$ for some i.

Suppose first that $G_{F_i} = 1$ for all i. In this case, we have $|G.F_i| = |G|$ for all i. This means that each orbit consists of $|G|$ components. In addition, any two components in the same orbit are homeomorphic and hence have the same Euler characteristic. Therefore, $|\Gamma/H| = |G|$ divides $\chi(\bar{M}^\alpha)$ and hence $\mathrm{Lef}(\alpha; \bar{M}, M^{S^1})$.

Now, consider the case in which $G_{F_i} \neq 1$ for some i. In this case, there exists non-trivial $[g] \in G$ such that $[g].F_i = F_i$ and hence $g.F_i = F_i$. Let $\pi : M \to \bar{M}$ denote the quotient map. Note that g acts isometrically on $\pi^{-1}(F_i)$. Since this action commutes with the circle action on $\pi^{-1}(F_i)$, g preserves some circle orbit C in $\pi^{-1}(F_i)$ (see [Ron05, Thm. A]). But α also acts on C and hence $\langle g, \alpha\rangle$ acts on C. Since this action is free, $\langle g, \alpha\rangle$ is cyclic. Since $g \notin \langle\alpha\rangle$, we conclude that $\langle\alpha\rangle$ is strictly contained in $\langle g, \alpha\rangle$, a contradiction. $\square$

Remark 4.3. In our applications, except in the first case of the proof of Lemma 5.5, the cohomology groups $H^i(\bar{M}, M^{S^1}; \mathbb{Q})$ have dimension at most one. Hence the induced action of α on $H^*(\bar{M}, M^{S^1}; \mathbb{Q})$ is trivial and we may replace $\mathrm{Lef}(\alpha;\bar{M},M^{S^1})$ by $\chi(\bar{M}, M^{S^1})$.

When applying Lemma 4.2, we need to figure out the cohomology groups $H^i(M/S^1, M^{S^1})$. The main tool in calculating these groups is the Smith-Gysin sequence which relates the relative cohomology groups $H^i(M/S^1, M^{S^1})$ to the cohomology groups of M and M^{S^1}.

Theorem 4.4 (Smith–Gysin sequence, [Bre72, p. 161])**.** *If S^1 acts on a paracompact space X, then there exists a long exact sequence*

$$\begin{aligned} \ldots \to H^i(X/S^1, X^{S^1}) \to H^i(X) \to H^{i-1}(X/S^1, X^{S^1}) \oplus H^i(X^{S^1}) \\ \to H^{i+1}(X/S^1, X^{S^1}) \to \ldots \end{aligned}$$

called the Smith–Gysin sequence. Here, we take coefficients in $\mathbb{Q}$.

5. Proof of Theorem A for rational cohomology Bazaikin spaces

In this section, we prove Theorem A for rational cohomology Bazaikin spaces. We equip the universal cover $\widetilde{M}$ of M with the pullback metric. We also lift the torus action to $\widetilde{M}$ (see [Bre72, Thm. I.9.1]) and then break the proof into subsections. Section 5.1 discusses the case in which the lifted torus action on $\widetilde{M}$ has non-empty fixed point set. In Section 5.2, we consider the case in which there is a circle inside T^2 whose action on $\widetilde{M}$ has a fixed point component of dimension one or three. In Section 5.3, we consider the case of a circle inside T^2 with five-dimensional fixed point set. Finally, in Section 5.4, we conclude the proof. Before proceeding to the proof, we prove the following lemma:

Lemma 5.1. *Let M^{13} be a closed, positively curved Riemannian manifold. If the universal cover of M is a rational cohomology Bazaikin space, then M does not have any totally geodesic submanifolds of codimension two or four.*

Proof. We proceed by contradiction. Let N be a totally geodesic submanifold of M. If the codimension of N equals two, then by Proposition 2.3, $\widetilde{N}$ and $\widetilde{M}$ are homotopy spheres and this contradicts the assumption that $\widetilde{M}$ is a rational cohomology Bazaikin space.

Suppose now that the codimension of N equals four. Theorem 2.1 implies that the inclusion $N \hookrightarrow M$ is 6-connected. Therefore, by Lemma 2.2, the homomorphism $\cup e : H^i(M;\mathbb{Z}) \to H^{i+4}(M;\mathbb{Z})$ is surjective for $3 \leq i < 6$ and injective for $3 < i \leq 6$. This implies that $H^5(M;\mathbb{Z}) \cong H^9(M;\mathbb{Z})$. Recall that $H^*(M;\mathbb{Q}) \cong H^*(\widetilde{M};\mathbb{Q})^\Gamma$, where $\Gamma = \pi_1(M)$ and where $H^*(\widetilde{M};\mathbb{Q})^\Gamma$ denotes the subring of elements invariant under the induced action of Γ (see [Bre72, Thm. III.2.4]). Since $H^5(\widetilde{M};\mathbb{Q}) = 0$, it follows that $H^5(M;\mathbb{Q}) = 0$. On the other hand, $H^9(\widetilde{M};\mathbb{Q}) \cong \mathbb{Q}$ and Γ acts trivially on $H^9(\widetilde{M};\mathbb{Q})$ by the proof of Lemma 3.2. Hence $H^9(M;\mathbb{Q}) \cong \mathbb{Q}$ and we have a contradiction. ☐

Notation 5.2. Throughout the rest of paper $\widetilde{M}^G$ (resp. M^G), where $G = S^1$ or T^2, denotes the fixed point set of the action of G on $\widetilde{M}$ (resp. M). Similarly, $\widetilde{M}_x^G$ (resp. M_x^G) denotes the component of $\widetilde{M}^G$ (resp. M^G) containing x.

5.1. Torus actions with fixed points

The first case in the proof of Theorem A is when the lifted torus action on $\widetilde{M}$ has a fixed point. In this case, we get a better bound for the index of cyclic subgroups of minimal index.

Lemma 5.3. *Let M^{13} be a closed Riemannian manifold with positive sectional curvature which admits an effective isometric T^2-action. Suppose that the universal cover of M is a rational cohomology Bazaikin space. If the lifted torus action on $\widetilde{M}$ has a fixed point, then $\pi_1(M)$ has a cyclic subgroup of index at most three.*

Note that Theorem A in the case of T^3 symmetry follows immediately since some $T^2 \subseteq T^3$ has a fixed point by Theorem 2.5.

Proof. Let x be a fixed point for the T^2-action on $\widetilde{M}$ and let $\mathrm{codim}(P \subseteq Q)$ denote the codimension of P in Q. Borel's formula (see [AP93, Thm. 5.3.11]) states that

$$\sum_{S^1 \subseteq T^2} \mathrm{codim}(\widetilde{M}_x^{T^2} \subseteq \widetilde{M}_x^{S^1}) = \mathrm{codim}(\widetilde{M}_x^{T^2} \subseteq \widetilde{M}),$$

where we take the sum over all the circles inside T^2. Note that only finitely many terms contribute to this sum. Note also that any fixed point component of an effective torus action on a positively curved manifold has even codimension. We break the proof into cases:

- $\dim(\widetilde{M}_x^{T^2}) = 1$. In this case, $\widetilde{M}_x^{T^2}$ is diffeomorphic to $\mathbb{S}^1$. Set $\Gamma = \pi_1(M)$ and let $\Gamma_2 \subseteq \Gamma$ be the subgroup of elements mapping the component $\widetilde{M}_x^{T^2}$ to itself. Note that Γ acts on $\widetilde{M}^{T^2}$ because its action on $\widetilde{M}$ commutes with the T^2-action. Since Γ_2 acts freely on $\widetilde{M}_x^{T^2}$ and $\widetilde{M}_x^{T^2}$ is a circle, Γ_2 is cyclic. In order to calculate the index of Γ_2, note that Γ acts on the set of components of $\widetilde{M}^{T^2}$ and hence

$$[\Gamma : \Gamma_2] \leq \#\{\text{components of } \widetilde{M}^{T^2}\}.$$

In addition (for the second inequality, see [AP93, Cor. 3.1.14]),

$$\#\{\text{components of } \widetilde{M}^{T^2}\} \leq \sum_i \dim H^{2i}(\widetilde{M}^{T^2};\mathbb{Q}) \leq \sum_i \dim H^{2i}(\widetilde{M};\mathbb{Q}) = 3.$$

Therefore, $[\Gamma : \Gamma_2] \leq 3$, as required.

- $\dim(\widetilde{M}_x^{T^2}) \geq 3$ and there exists $S^1 \subseteq T^2$ with $\mathrm{codim}(\widetilde{M}_x^{T^2} \subseteq \widetilde{M}_x^{S^1}) = 2$. Let Γ_2 be the subgroup of Γ which acts on $\widetilde{M}_x^{T^2}$. As argued in the previous case, the index of Γ_2 in Γ is at most three. Since Γ_2 acts on $\widetilde{M}_x^{T^2}$, it also acts on $\widetilde{M}_x^{S^1}$. Proposition 2.4 now implies that Γ_2 is cyclic.

• $\dim(\widetilde{M}_x^{T^2}) \geq 3$ and $\operatorname{codim}(\widetilde{M}_x^{T^2} \subseteq \widetilde{M}_x^{S^1}) \neq 2$ for all $S^1 \subseteq T^2$. By Lemma 5.1, the dimension of $\widetilde{M}_x^{S^1}$ is at most seven. Therefore,

$$7 \leq \dim(\widetilde{M}_x^{T^2}) + 4 \leq \dim(\widetilde{M}_x^{T^2}) + \operatorname{codim}(\widetilde{M}_x^{T^2} \subseteq \widetilde{M}_x^{S^1}) = \dim(\widetilde{M}_x^{S^1}) \leq 7$$

for all $S^1 \subseteq T^2$ with $\operatorname{codim}(\widetilde{M}_x^{T^2} \subseteq \widetilde{M}_x^{S^1}) > 0$. Hence equality holds and we have $\dim(\widetilde{M}_x^{T^2}) = 3$ and $\dim(\widetilde{M}_x^{S^1}) = 7$. On the one hand, the right-hand side of Borel's formula equals 10. On the other hand, the left-hand side is a multiple of four. This is a contradiction. $\square$

5.2. Fixed point component of dimension one or three

In the presence of T^2 symmetry, Theorem 2.5 guarantees existence of a circle whose fixed point set is non-empty. In this section, we prove Theorem A in the case where this fixed point set has a component of dimension one or three.

Lemma 5.4. *Let M^{13} and T^2 be as in Theorem A. If there exists $S^1 \subseteq T^2$ such that $\widetilde{M}^{S^1}$ has a component $\widetilde{M}_x^{S^1}$ of dimension one or three, then $\Gamma := \pi_1(M)$ has a cyclic subgroup of index dividing six.*

Proof. Let $\Gamma_1 \subseteq \Gamma$ be the subgroup that acts trivially on $H^*(\widetilde{M};\mathbb{Q})$. Recall by the proof of Lemma 3.2 that Γ_1 has index at most two. By Theorem 2.7, we have $\Gamma_1 \cong \mathbb{Z}_{2^a} \times \Gamma_1'$ for some $a \geq 0$ and some group Γ_1' of odd order. Next, let $\Gamma_2' \subseteq \Gamma_1'$ be the subgroup preserving the component $\widetilde{M}_x^{S^1}$. As in the proof of Lemma 5.3, we see that $\widetilde{M}^{S^1}$ has at most three components and therefore that $\Gamma_2' \subseteq \Gamma_1'$ has index at most three. Note in addition that the index is a divisor of three because Γ_1' has odd order. We claim that Γ_2' is cyclic. In order to prove this, we consider each case separately.

• $\dim(\widetilde{M}_x^{S^1}) = 1$. Since $\widetilde{M}_x^{S^1}$ is diffeomorphic to $\mathbb{S}^1$ and the action of Γ_2' on $\widetilde{M}_x^{S^1}$ is free, Γ_2' is cyclic.

• $\dim(\widetilde{M}_x^{S^1}) = 3$. Let $N := \widetilde{M}_x^{S^1}$. By Lemma 5.3, we may assume that $\widetilde{M}^{T^2} = \varnothing$ and hence that there exists $S_1^1 \subseteq T^2$ such that $N^{S_1^1} = \varnothing$. We apply Lemma 4.2 to the action of S_1^1 on N. For this, we use the Smith-Gysin sequence to calculate $\chi(\bar{N}, N^{S_1^1}) = \chi(\bar{N}, \varnothing) = 2$. Since Γ_2' has odd order, Lemma 4.2 (see also Remark 4.3) implies that Γ_2' does not contain a copy of $\mathbb{Z}_p \times \mathbb{Z}_p$ for any p. By Theorem 2.10, every Sylow subgroup of Γ_2' is cyclic. Hence by Theorem 2.11, $\Gamma_2' \in \mathcal{C}_d$ for some $d \geq 1$. Lemma 4.2 then implies that $d = 1$ and hence that Γ_2' is cyclic by Remark 2.13.

Therefore, Γ_1', and hence Γ_1, has a cyclic subgroup of index dividing three. Since $[\Gamma : \Gamma_1] \leq 2$, we are done. $\square$

5.3. Five-dimensional fixed point set

In this section, we assume that there exists some $S^1 \subseteq T^2$ whose fixed point set is five-dimensional. Note that by Lemma 5.4, we only need to discuss the case in which all components of $\widetilde{M}^{S^1}$ are five-dimensional. Since each component of

$\widetilde{M}^{S^1}$ is a totally geodesic and hence positively curved submanifold of $\widetilde{M}$, the first rational Betti number of each component of the fixed point set $\widetilde{M}^{S^1}$ is zero. In addition, by Thereom 2.6, the sum of the rational Betti numbers of $\widetilde{M}^{S^1}$ is at most six. Moreover, $\widetilde{M}^{S^1}$ cannot have the same rational cohomology ring as $(\mathbb{C}\mathrm{P}^1 \times \mathbb{S}^3)\#(\mathbb{C}\mathrm{P}^1 \times \mathbb{S}^3)$ since otherwise we will have $\sum_i b_i(\widetilde{M}^{S^1};\mathbb{Q}) = \sum_i b_i(\widetilde{M};\mathbb{Q})$. But this contradicts Theorem 2.6 because the rational cohomology ring of $(\mathbb{C}\mathrm{P}^1 \times \mathbb{S}^3)\#(\mathbb{C}\mathrm{P}^1 \times \mathbb{S}^3)$ has four generators. Altogether, we get that each component of $\widetilde{M}^{S^1}$ has the rational cohomology of either $\mathbb{S}^5$ or $\mathbb{C}\mathrm{P}^1 \times \mathbb{S}^3$. Therefore, the rational cohomology ring of $\widetilde{M}^{S^1}$ must be the same as that of one of the following spaces:

$$\underbrace{\mathbb{S}^5 \sqcup \ldots \sqcup \mathbb{S}^5}_{\text{k times}} \ (1 \le k \le 3), \quad \mathbb{C}\mathrm{P}^1 \times \mathbb{S}^3, \quad \mathbb{S}^5 \sqcup (\mathbb{C}\mathrm{P}^1 \times \mathbb{S}^3). \tag{1}$$

Lemma 5.5. *For M^{13} and T^2 as in Theorem A, if there exists $S^1 \subseteq T^2$ such that the fixed point set $\widetilde{M}^{S^1}$ is five-dimensional, then $\Gamma := \pi_1(M)$ has a cyclic subgroup of index dividing* 18 *or* 27.

Proof. By the discussion before Lemma 5.5, we may assume that $\widetilde{M}^{S^1}$ has the same rational cohomology as one of the five spaces in (1). We break the proof into cases:

• $\widetilde{M}^{S^1}$ has a component N_1 with the rational cohomology of $\mathbb{C}\mathrm{P}^1 \times \mathbb{S}^3$. As in Lemma 5.4, let $\Gamma_1 \subseteq \Gamma$ be the subgroup of index at most two that acts trivially on $H^*(\widetilde{M};\mathbb{Q})$ and write Γ_1 as $\Gamma_1 \cong \mathbb{Z}_{2^a} \times \Gamma_1'$ for some $a \ge 0$ and some group Γ_1' of odd order. Since the action of Γ_1' on $\widetilde{M}$ commutes with the S^1-action, Γ_1' acts on $\widetilde{M}^{S^1}$. Since $\widetilde{M}^{S^1}$ has at most one component with the rational cohomology of $\mathbb{C}\mathrm{P}^1 \times \mathbb{S}^3$, Γ_1' acts on N_1. By Lemma 5.3, we may assume that $\widetilde{M}^{T^2} = \varnothing$ and hence that there exists $S^1_2 \subseteq T^2$ such that $N_1^{S^1_2} = \varnothing$. Let $\bar{N}_1 := N_1/S^1_2$. By applying the Smith-Gysin sequence to the pair $(\bar{N}_1, N_1^{S^1_2}) = (\bar{N}_1, \varnothing)$, it follows that $H^i(\bar{N}_1;\mathbb{Q})$ is isomorphic to $\mathbb{Q}$ for $i \in \{0,4\}$, isomorphic to $\mathbb{Q}^2$ for $i = 2$, and trivial otherwise.

Let $\alpha \in \Gamma_1'$. We claim that $\mathrm{Lef}(\alpha;\bar{N}_1)$ equals 1 or 4. In order to prove this, note that since α has odd order, the induced action of α on $H^i(\bar{N}_1;\mathbb{Q})$ is trivial for $i \in \{0,4\}$. Now, let λ_1 and λ_2 denote the eigenvalues of $\alpha^* : H^2(\bar{N}_1;\mathbb{Q}) \to H^2(\bar{N}_1;\mathbb{Q})$. Since α has finite order, λ_1 and λ_2 are roots of unity, so $\lambda_1 + \lambda_2 \in [-2,2]$. Moreover, the Lefschetz number and hence $\lambda_1+\lambda_2$ is an integer. Therefore, $\lambda_1+\lambda_2 \in \{-2,-1,0,1,2\}$. In addition, λ_1 and λ_2 have odd orders since α has odd order. In particular, $\lambda_i \ne -1$, so $\lambda_1 + \lambda_2 \ne -2$. Similarly, if $\lambda_1 \ne 1$, then it is complex and hence $\lambda_2 = \bar{\lambda}_1$ and $\lambda_1 + \lambda_2 = 2Re(\lambda_1)$. Therefore, if $\lambda_1 + \lambda_2 \in \{0,1\}$, then λ_1 has order 4 or 6, a contradiction. Hence $\lambda_1 + \lambda_2 \in \{-1,2\}$. This proves the claim.

Lemma 4.2 now implies that Γ_1' does not contain $\mathbb{Z}_p \times \mathbb{Z}_p$ for all p. Hence by Theorem 2.10, all Sylow subgroups of Γ_1' are cyclic. Theorem 2.11 and Lemma 4.2 then imply that $\Gamma_1' \in \mathcal{C}_d$, where $d = 1$. Therefore, Γ_1', and hence Γ_1, is cyclic. This means that Γ has a cyclic subgroup of index at most two.

• $\widetilde{M}^{S^1}$ is a rational cohomology 5-sphere. Let $N_2 := \widetilde{M}^{S^1}$. Note that Γ acts on N_2. Moreover, Γ acts trivially on the rational cohomology of N_2 by Weinstein's theorem. Therefore, we can apply Theorem 2.7 to conclude that $\Gamma \cong \mathbb{Z}_{2^b} \times \Gamma'$, where $b \geq 0$ and Γ' is a group of odd order. As in the previous case, we can choose $S^1_3 \subseteq T^2$ such that $N_2^{S^1_3} = \varnothing$ and apply the Smith-Gysin sequence to the pair $(\bar{N}_2, N_2^{S^1_3}) = (\bar{N}_2, \varnothing)$ to get $\chi(\bar{N}_2) = 3$. Lemma 4.2 then implies that Γ' does not contain a copy of $\mathbb{Z}_p \times \mathbb{Z}_p$ for $p \neq 3$. This, together with Theorem 2.10, implies that all Sylow p-subgroups of Γ' are cyclic for $p \neq 3$. Now, we claim that the Sylow 3-subgroup of Γ' is either cyclic or isomorphic to one of $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$. The idea is to prove that Γ' does not contain a copy of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $U(3,3)$, or $\mathbb{Z}_9 \times \mathbb{Z}_3$. Proposition 2.14 then implies the claim. By Lemma 4.2, in order to prove that Γ' does not contain a copy of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $U(3,3)$, or $\mathbb{Z}_9 \times \mathbb{Z}_3$, it suffices to find a normal subgroup of order three of each of these groups which is not strictly contained in a cyclic subgroup. Existence of such a subgroup is obvious in the case of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_9 \times \mathbb{Z}_3$. As for $U(3,3)$, it has center isomorphic to $\mathbb{Z}_3$. This subgroup is normal and is not strictly contained in a larger cyclic subgroup because every non-trivial element of $U(3,3)$ has order three.

Now, we have a group Γ' of odd order such that its Sylow p-subgroups are cyclic for $p \neq 3$ and its Sylow 3-subgroup is either cyclic, $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$. Let P_3 be a Sylow 3-subgroup of Γ'. If P_3 is cyclic, then $\Gamma' \in \mathcal{C}_d$ for some $d \geq 1$ by Theorem 2.11. If not, then we apply Theorem 2.15. Hence Γ' can be written in the form of $P_3 N$ for some $N \trianglelefteq \Gamma'$ such that $P_3 \cap N = \{1\}$. Letting $\Gamma_2'' := \mathbb{Z}_3 N$ or $\Gamma_2'' := \mathbb{Z}_9 N$, depending on whether $P_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $P_3 \cong \mathbb{Z}_9 \rtimes \mathbb{Z}_3$,we get an index three subgroup Γ_2'' of Γ' such that all Sylow subgroups of Γ_2'' are cyclic. By Theorem 2.11, it follows that $\Gamma_2'' \in \mathcal{C}_d$ for some d. The discussion above shows that either Γ' or Γ_2'' has a cyclic subgroup of index d. Now, Lemma 4.2 implies that d divides three. Therefore, Γ', and hence Γ, has a cyclic subgroup of index dividing nine, as required.

- $\widetilde{M}^{S^1}$ is the disjoint union of two rational cohomology 5-spheres. In this case, let $\widetilde{M}^{S^1}_x$ denote one of the two components of $\widetilde{M}^{S^1}$ and let Γ_3 be the subgroup of Γ of index at most two which acts on $\widetilde{M}^{S^1}_x$. Replacing Γ by Γ_3 in the argument for the previous case, it follows that Γ has a cyclic subgroup of index dividing 18.
- $\widetilde{M}^{S^1}$ is the disjoint union of three rational cohomology 5-spheres. Let $\widetilde{M}^{S^1}_x$ denote one of the components of $\widetilde{M}^{S^1}$ and let $\Gamma_3 \subseteq \Gamma$ be the subgroup of index at most three that acts on $\widetilde{M}^{S^1}_x$. We again argue as in the second case, replacing Γ by Γ_3 to conclude that Γ has a cyclic subgroup of index dividing 18 or 27. □

5.4. Conclusion of proof

Let M and T^2 be as in Theorem A. As remarked at the beginning of this section, we may lift the T^2-action to the universal cover of M. By Theorem 2.5, we may choose $S^1 \subseteq T^2$ such that $\widetilde{M}^{S^1} \neq \varnothing$. If $\widetilde{M}^{S^1}$ has a component of dimension one or three, then we are done by Lemma 5.4. In particular, we are done if $\dim(\widetilde{M}^{S^1}) \leq 3$. In addition, if $\dim(\widetilde{M}^{S^1}) = 5$, then the proof follows by Lemma 5.5. Recall also that the dimension of $\widetilde{M}^{S^1}$ is at most seven by Lemma 5.1. We may assume therefore that $\dim(\widetilde{M}^{S^1}) = 7$.

Note that by Frankel's theorem (see [Fra61]), $\widetilde{M}^{S^1}$ cannot have more than one component of dimension seven. Moreover, the seven-dimensional component $\widetilde{M}^{S^1}_x$ of $\widetilde{M}^{S^1}$ satisfies $b_2(\widetilde{M}^{S^1}_x;\mathbb{Q}) = 1$ since the inclusion $\widetilde{M}^{S^1}_x \hookrightarrow \widetilde{M}$ is 3-connected by

Theorem 2.1. By Poincaré duality, $\widetilde{M}_x^{S^1}$ has total Betti number at least four. Suppose for a moment that $\widetilde{M}^{S^1}$ is not connected. Let $\widetilde{M}_y^{S^1}$ denote another component. By Theorem 2.6, $\widetilde{M}_y^{S^1}$ is a rational $\mathbb{S}^1$, $\mathbb{S}^3$, or $\mathbb{S}^5$ because $\widetilde{M}$ has total Betti number six. It follows by Lemma 5.4 and by the second case in the proof Lemma 5.5 that $\pi_1(M)$ has a cyclic subgroup of index 1, 2, 3, 6, or 9. Therefore, we may assume that $\widetilde{M}^{S^1}$ is connected.

Lemma 5.6. *Suppose that T^2 acts effectively and isometrically on a closed, positively curved manifold M^{13} whose universal cover is a rational cohomology Bazaikin space. If there exists $S^1 \subseteq T^2$ such that $\widetilde{M}^{S^1}$ is connected and seven-dimensional, then at least one of the following holds:*

(1) *$\pi_1(M)$ is cyclic.*
(2) *$\widetilde{M}^{T^2} \neq \varnothing$.*
(3) *There exists another circle $S_1^1 \subseteq T^2$ such that $\widetilde{M}^{S_1^1}$ is non-empty and has dimension at most five.*

Proof. Let $p : \widetilde{M} \to M$ be the universal covering map. Note that $p(\widetilde{M}^{S^1}) = M^{S^1}$, so M^{S^1} is connected and seven-dimensional.

If the action of S^1 on M does not have any non-trivial finite isotropy groups, then $\pi_1(M)$ is cyclic (see [Ron05, Thm. C]) and we are done. Hence we may assume that this action has a non-trivial finite isotropy group S_q^1. Let $N := M_q^{S_q^1}$.

We claim that $N \cap M^{S^1}$ is empty. Indeed, if $q' \in N \cap M^{S^1}$, then we have $M^{S^1} \subseteq M_{q'}^{S_q^1} = M_q^{S_q^1}$. Now, $q \notin M^{S^1}$ since the isotropy group S_q^1 is finite, so this inclusion is strict. But then the dimension of $M_q^{S_q^1}$ would be greater than seven, so we have a contradiction to Lemma 5.1.

Next we claim that there exists a circle $S_1^1 \subseteq T^2$ such that $N^{S_1^1} \neq \varnothing$. In order to prove this, note that T^2 acts on N. Let H denote the kernel of this action. If H is not finite, then there exists $S_1^1 \subseteq H$ and we are done. If H is finite, then the claim follows by applying Theorem 2.5 to the action of $T^2/H \cong T^2$ on N. Therefore, we can choose $q_1 \in N^{S_1^1}$.

We claim that $S_1^1 \neq S^1$. If instead $S_1^1 = S^1$, then $q_1 \in M^{S^1}$ since q_1 is fixed by S_1^1. But $q_1 \in N$, so $q_1 \in N \cap M^{S^1}$, a contradiction.

In order to conclude the proof, we consider two cases. If $\dim(M^{S_1^1}) \leq 5$, then $\dim(\widetilde{M}^{S_1^1}) \leq 5$ and we are done. If not, then $M^{S^1} \cap M^{S_1^1} \neq \varnothing$ by Frankel's theorem. In this case, T^2 has a fixed point on M and hence on $\widetilde{M}$. $\square$

Lemma 5.6, together with Lemmas 5.3, 5.4, and 5.5, completes the proof of Theorem A for rational cohomology Bazaikin spaces.

6. Proof of Theorem A for mod 3 cohomology Bazaikin spaces

In this section, we prove Theorem A for mod 3 cohomology Bazaikin spaces. The idea here is to find a subgroup $\Gamma_3 \subseteq \Gamma$ of index at most three which does not contain a copy of $\mathbb{Z}_p \times \mathbb{Z}_p$ for any p. Recall that by a result of Smith, $\mathbb{Z}_p \times \mathbb{Z}_p$ cannot act freely on a mod p cohomology sphere (see [Smi44]). Heller proved that

$\mathbb{Z}_p^3$ cannot act freely on a manifold M with $H^*(M;\mathbb{Z}_p) \cong H^*(\mathbb{S}^m \times \mathbb{S}^n;\mathbb{Z}_p)$, where $n > m$ (see [Hel59], cf. [AP93, Example 3.10.17]). We refine Heller's argument in a special case to prove the following lemma:

Lemma 6.1. *If M is a smooth manifold such that $H^*(M;\mathbb{Z}_p) \cong H^*(\mathbb{S}^2 \times \mathbb{S}^3;\mathbb{Z}_p)$ for some odd prime p, then $\mathbb{Z}_p \times \mathbb{Z}_p$ cannot act freely on M.*

Proof. Suppose by contradiction that $G = \mathbb{Z}_p \times \mathbb{Z}_p$ acts freely on M and consider the Leray-Serre spectral sequence associated to the Borel fibration $M \to M_G \to BG$. Note that since p is odd and $\dim H^i(M;\mathbb{Z}_p) \leq 1$ for all i, G acts trivially on $H^*(M;\mathbb{Z}_p)$. Therefore, we can apply the Leray–Serre spectral sequence.

Since G acts freely on M, it follows from [AP93, Prop. 3.10.9] that $H^*(M_G;\mathbb{Z}_p) \cong H^*(M/G;\mathbb{Z}_p)$. By [AP93, Lem. 3.10.16], this implies that $H^i(M_G;\mathbb{Z}_p) = 0$ for all $i > 5$. In particular, $H^6(M_G;\mathbb{Z}_p) = 0$ and hence $E_\infty^{6,0}$ is trivial. Observe that for the differential map $d_r : E_r^{m,n} \to E_r^{6,0}$, m and n satisfy $m + n = 5$ and $n \geq 1$. Moreover, $E_2^{m,n} = 0$ for $n = 1$ or 4. Hence the only non-trivial differentials which can kill elements of $E_2^{6,0}$ come from $E_3^{3,2}$, $E_4^{2,3}$ and $E_6^{0,5}$. Note that $H^*(BG;\mathbb{Z}_p) = \mathbb{Z}_p[t_1, t_2] \otimes \wedge(s_1, s_2)$, where each t_i has degree two, each s_i has degree one, and $\wedge(s_1, s_2)$ denotes the exterior algebra over $\mathbb{Z}_p$ generated by s_1 and s_2. Therefore, the set $\{t_1^i t_2^j : i + j = 3\} \cup \{s_1 s_2 t_1^i t_2^j : i + j = 2\}$ forms a basis for $E_2^{6,0}$ and hence $\dim E_2^{6,0} = 7$. Moreover,

$$\dim E_2^{0,5} + \dim E_2^{2,3} + \dim E_2^{3,2} = \sum_{\substack{0 \leq i \leq 3 \\ i \neq 1}} \dim H^i(BG;\mathbb{Z}_p) = 1 + 3 + 4 = 8.$$

We get a contradiction by proving that $E_\infty^{6,0}$ cannot be trivial. By considering the dimensions calculated above, in order to prove this, it suffices to find at least two basis elements in the groups $E_2^{3,2}$, $E_2^{2,3}$, and $E_2^{0,5}$ that either do not survive to the appropriate page or do not kill any elements of $E_2^{6,0}$. In order to find such basis elements, we need to analyze the differentials. Fix generators $x \in H^3(M;\mathbb{Z}_p)$ and $y \in H^2(M;\mathbb{Z}_p)$. We break the proof into two cases:

- The differential $d_2 : E_2^{1,3} \to E_2^{3,2}$ is non-zero. In this case, the differential $d_2 : E_2^{0,3} \to E_2^{2,2}$ is non-zero because $\{s_1 x, s_2 x\}$ forms a basis for $E_2^{1,3}$ and $d_2(s_i) = 0$. Note that $d_2(x)$ is of the form $a_1 t_1 y + a_2 t_2 y + a_3 s_1 s_2 y$ for some $a_i \in \mathbb{Z}_p$. Note also that at least one of a_1 or a_2 is non-zero since otherwise d_2 vanishes on $E_2^{1,3}$. Therefore, $d_2(s_i x) = -a_1 s_i t_1 y - a_2 s_i t_2 y$ and hence $d_2 : E_2^{1,3} \to E_2^{3,2}$ is injective. This means that two of the basis elements of $E_2^{3,2}$ do not survive to the E_3 page and hence cannot kill any elements of $E_3^{6,0}$.
- The differential $d_2 : E_2^{1,3} \to E_2^{3,2}$ is zero. In this case, $E_3^{3,2} \cong E_2^{3,2} \cong H^3(BG;\mathbb{Z}_p)$. Since y survives to the E_3 page, the set $\{\bar{s}_i \bar{t}_j \bar{y} : i, j \in \{1, 2\}\}$ forms a basis for $E_3^{3,2}$, where the bars denote the images of elements from the E_2 page in the E_3 page. In addition, $d_3(\bar{y}) = b_1 \bar{s}_1 \bar{t}_1 + b_2 \bar{s}_1 \bar{t}_2 + b_3 \bar{s}_2 \bar{t}_1 + b_4 \bar{s}_2 \bar{t}_2$ for some $b_i \in \mathbb{Z}_p$, and each differential map $d_r : E_r^{m,n} \to E_r^{m+r,n-r+1}$ is a derivation. Therefore, the differential $d_3 : \mathbb{Z}_p^4 \cong E_3^{3,2} \to E_3^{6,0} \cong \mathbb{Z}_p^7$ has its image in the 3-dimensional subspace of $E_3^{6,0}$ spanned by $\{\bar{s}_1 \bar{s}_2 \bar{t}_1^i \bar{t}_2^j : i + j = 2\}$. But any linear map from a 4-dimensional space to a 3-dimensional space has non-trivial kernel. Hence the

kernel of $d_3 : E_3^{3,2} \to E_3^{6,0}$ is at least 1-dimensional and so not all basis elements of $E_3^{3,2}$ can contribute to killing elements of $E_3^{6,0}$. Now, we only need to find one more basis element in another group that does not kill any elements of $E_2^{6,0}$.

Without loss of generality, we may assume that $d_3(\bar{y}) \neq 0$ since otherwise $d_3 : E_3^{3,2} \to E_3^{6,0}$ would be trivial and so $E_3^{3,2}$ cannot kill any elements of $E_3^{6,0}$. If $d_2(x) = 0$, then x survives to the E_3 page and so we have $d_3(\bar{x}\bar{y}) = d_3(\bar{x})\bar{y} - \bar{x}d_3(\bar{y}) \neq 0$. This means that $E_3^{0,5}$ does not survive to the E_4 page and hence it cannot kill any elements of $E_6^{6,0}$. If instead $d_2(x) \neq 0$, then $d_2 : E_2^{2,3} \to E_2^{4,2}$ would be non-zero. This means that at least one of the basis elements of $E_2^{2,3}$ does not survive to the E_3 page and so it cannot kill any elements of $E_4^{6,0}$. $\square$

We now proceed to the proof of Theorem A. As the proof for rational cohomology Bazaikin spaces shows, index 18 or 27 would possibly arise only when the fixed point set of the circle action is the disjoint union of either two or three rational cohomology 5-spheres. Hence we only need to discuss those two cases.

First, suppose that $\widetilde{M}^{S^1} = \widetilde{M}_x^{S^1} \sqcup \widetilde{M}_y^{S^1}$, where each component is a rational cohomology 5-sphere. Consider $\mathbb{Z}_3 \subseteq S^1$. We may assume that $\widetilde{M}_x^{S^1} = \widetilde{M}_x^{\mathbb{Z}_3}$ since otherwise we get strict inclusions $\widetilde{M}_x^{S^1} \subsetneq \widetilde{M}_x^{\mathbb{Z}_3} \subsetneq \widetilde{M}$. Lemma 5.1 then implies that $\dim(\widetilde{M}_x^{\mathbb{Z}_3}) = 7$. By Proposition 2.3, $\widetilde{M}_x^{\mathbb{Z}_3}$ is a homotopy sphere. Since the inclusion $\widetilde{M}_x^{\mathbb{Z}_3} \hookrightarrow \widetilde{M}$ is 2-connected by Theorem 2.1, we have a contradiction. Similarly, $\widetilde{M}_y^{S^1} = \widetilde{M}_y^{\mathbb{Z}_3}$. Moreover,

$$\sum_i \dim H^i(\widetilde{M}^{\mathbb{Z}_3}; \mathbb{Z}_3) \leq \sum_i \dim H^i(\widetilde{M}; \mathbb{Z}_3) \leq 10.$$

Together with Poincaré duality, this implies that at least one of the fixed point set components, say $\widetilde{M}_x^{S^1}$, has the mod 3 cohomology of either $\mathbb{S}^5$ or $\mathbb{S}^2 \times \mathbb{S}^3$ (note that $\widetilde{M}_x^{S^1}$ is positively curved, so it has vanishing first integral Betti number. This, together with the universal coefficient theorem, implies that $\widetilde{M}_x^{S^1}$ cannot have the mod 3 cohomology of $\mathbb{S}^1 \times \mathbb{S}^4$). Let Γ_3 denote the subgroup of Γ of index at most two which acts on $\widetilde{M}_x^{S^1}$. By Theorem 2.7, we have $\Gamma_3 \cong \mathbb{Z}_{2^c} \times \Gamma_3'$, where $c \geq 0$ and Γ_3' is a group of odd order. As the proof of Theorem A for rational cohomology Bazaikin spaces shows, Γ_3' does not contain a copy of $\mathbb{Z}_p \times \mathbb{Z}_p$ for $p \neq 3$. In addition, Lemma 6.1, together with the fact that $\mathbb{Z}_p \times \mathbb{Z}_p$ cannot act freely on a mod p cohomology sphere, implies that Γ_3' does not contain $\mathbb{Z}_3 \times \mathbb{Z}_3$. Hence all Sylow subgroups of Γ_3' are cyclic and $\Gamma_3' \in \mathcal{C}_d$ for some $d \geq 1$. Our calculations from the previous section imply that d divides three. This means that Γ_3 has a cyclic subgroup of index dividing three and hence Γ has a cyclic subgroup of index dividing six.

The proof for the case where $\widetilde{M}^{S^1}$ is the disjoint union of three rational cohomology 5-spheres is similar except that here the bound on $\dim H^*(\widetilde{M}^{\mathbb{Z}_3}; \mathbb{Z}_3)$ implies that at least one of the components of $\widetilde{M}^{S^1}$ is a mod 3 cohomology $\mathbb{S}^5$. Moreover, in this case, the index of Γ_3 in Γ is at most three and hence we get that Γ has a cyclic subgroup of index $1, 2, 3, 6$, or 9.

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