Positively curved manifolds with discrete symmetry

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Setup and background

Setup: M – closed, Riemannian, sec > 0.

- $\dim(M) = 2 \Longrightarrow M \simeq \mathbb{S}^2$ (Gauss-Bonnet, classification of surfaces).
- $\dim(M) = 3 \Longrightarrow M \simeq \mathbb{S}^3/\Gamma$ (Hamilton).
- $\dim(M) \geq 4$? (open problem).
- Simply connected examples in dimension > 24: $\mathbb{S}^n, \mathbb{C}\mathrm{P}^n, \mathbb{H}\mathrm{P}^n$.

Idea: Study manifolds with symmetry.

- $T^r \subseteq \mathrm{Isom}(M, g)$.

Maximal torus and $\mathbb{Z}_2$-torus symmetry

Setup: M – closed, Riemannian, sec > 0.

Theorem (Grove-Searle, 1994): $T^r \subseteq Isom(M^n, g) \Longrightarrow r \leq \left\lfloor \frac{n+1}{2} \right\rfloor$, with equality only if M is diffeomorphic to $\mathbb{S}^n, \mathbb{RP}^n, \mathbb{CP}^{\frac{n}{2}}$, or a lens space.

Theorem (Fang-Grove, 2016): $\mathbb{Z}_2^r \subseteq \mathrm{Isom}(M^n, g)$, non-empty fixed point set $\Longrightarrow r \leq n$, with equality only if M diffeomorphic to $\mathbb{S}^n$ or $\mathbb{RP}^n$.

Example

- $G := \mathbb{Z}_2^{n+1} \subseteq \mathrm{O}(n+1) \cong \mathrm{Isom}(\mathbb{S}^n), e_1 \in \mathbb{S}^n \Longrightarrow G_{e_1} \cong \mathbb{Z}_2^n$.
- $\mathrm{Isom}(\mathbb{RP}^n) \cong \mathrm{O}(n+1)/\mathbb{Z}_2 \Longrightarrow \mathbb{Z}_2^n \subseteq \mathrm{Isom}(\mathbb{RP}^n)$.

Half-maximal torus and $\mathbb{Z}_2$-torus symmetry

Setup: M – closed, Riemannian, sec > 0.

Theorem (Wilking, 2003): If $T^r \subseteq Isom(M^n, g)$, $n \geq 8$, $r \geq \frac{n}{4} + 1$, then

1. M homotopy equivalent to $\mathbb{S}^n, \mathbb{RP}^n, \mathbb{CP}^{\frac{n}{2}}$, or a lens space; or
2. M homeomorphic to $\mathbb{HP}^{\frac{n}{4}}$ and $r = \frac{n}{4} + 1$.

Theorem (K.K.S.): If $\mathbb{Z}_2^r \subseteq \mathrm{Isom}(M^n, g)$, non-empty fixed point set, $n > 12$, $r \geq \frac{n}{2} + 1$, then

1. M homotopy equivalent to $\mathbb{S}^n$ or $\mathbb{RP}^n$; or
2. M homotopy equivalent to $\mathbb{CP}^{\frac{n}{2}}$ and $r = \frac{n}{2} + 1$.

Example (half-maximal $\mathbb{Z}_2$-torus symmetry)

Setup: M – closed, Riemannian, sec > 0.

Theorem (K.K.S.): $\mathbb{Z}_2^r \subseteq \mathrm{Isom}(M^n, g)$, with fixed point, $r \geq \frac{n}{2} + 1$.

1. M homotopy equivalent to $\mathbb{S}^n$ or $\mathbb{RP}^n$; or
2. M homotopy equivalent to $\mathbb{CP}^{\frac{n}{2}}$ and $r = \frac{n}{2} + 1$.

Example

$\mathrm{Isom}(\mathbb{CP}^{r-1}) \cong \mathrm{PU}(r) \rtimes \mathbb{Z}_2$ $([z_0, z_1, \ldots, z_{r-1}] \mapsto [\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_{r-1}])$.

- $\Delta\mathbb{Z}_2 :=$ the diagonal subgroup of $\mathbb{Z}_2^r \subseteq \mathrm{U}(r)$.
- $\mathbb{Z}_2^{r-1} \cong \mathbb{Z}_2^r / \Delta\mathbb{Z}_2$.
- $\mathbb{Z}_2^r \cong \mathbb{Z}_2^{r-1} \times \mathbb{Z}_2 \subseteq \mathrm{Isom}(\mathbb{CP}^{r-1})$.

Non-example (half-maximal $\mathbb{Z}_2$-torus symmetry)

Setup: M – closed, Riemannian, sec > 0.

Theorem (K.K.S.): $\mathbb{Z}_2^r \subseteq \mathrm{Isom}(M^n, g)$, with fixed point, $r \geq \frac{n}{2} + 1$.

1. M homotopy equivalent to $\mathbb{S}^n$ or $\mathbb{RP}^n$; or
2. M homotopy equivalent to $\mathbb{CP}^{\frac{n}{2}}$ and $r = \frac{n}{2} + 1$.

Example

$\mathrm{Isom}(\mathbb{S}^{2k-1}) \supseteq \mathrm{U}(k) \rtimes \mathbb{Z}_2$ $((z_1, \ldots, z_k) \mapsto (\bar{z}_1, \ldots, \bar{z}_k))$.

- $\mathbb{Z}_2^k \subseteq \mathrm{U}(k)$ descends to $\mathbb{S}^{2k-1}/\mathbb{Z}_m$.
- $\mathbb{Z}_2$ descends to $\mathbb{S}^{2k-1}/\mathbb{Z}_m$.
- $\mathbb{Z}_2^{\frac{2k-1}{2}+0.5} \cong \mathbb{Z}_2^{k-1} \times \mathbb{Z}_2 \subseteq \mathrm{Isom}(\mathbb{S}^{2k-1}/\mathbb{Z}_m)$ (n=2k-1).

Quarter-maximal torus and $\mathbb{Z}_2$-torus symmetry

Setup: M^n – closed, Riemannian, sec > 0, n even.

Theorem (Rong-Su, 2005): If $T^r \subseteq Isom(M^n, g)$, $n > 12$, $r \geq \frac{n-4}{8}$, then

1. For any component F of the fixed point set M^{T^r}, $\chi(F) > 0$;
2. Consequently, $\chi(M) > 0$.

Theorem (K.K.S.): If $\mathbb{Z}_2^r \subseteq \text{Isom}(M^n, g)$ with fixed point p, $n > 12$, $r \geq \frac{n}{4} + 2$, then

1. For every $\mathbb{Z}_2^{r-4} \subseteq \mathbb{Z}_2^r$, the fixed point component of $\mathbb{Z}_2^{r-4}$ containing p is homotopy equivalent to a sphere, a real projective space, a complex projective space, or a lens space; or
2. M is an integeral cohomology $\mathbb{HP}^{\frac{n}{4}}$ and $r = \frac{n}{4} + 2$.

Corollary (K.K.S.): Every component F of the fixed point set $M^{\mathbb{Z}_2^r}$ is homotopy equivalent to a lens space, or complex projective space. In particular, $\chi(F) > 0$ if $\dim(F)$ is even.

Example (quarter-maximal $\mathbb{Z}_2$-torus symmetry)

Setup: M – closed, Riemannian, sec > 0.

Theorem (K.K.S.): $\mathbb{Z}_2^r \subseteq \text{Isom}(M^n, g)$, with fixed point, $r \geq \frac{n}{4} + 2$.

1. For every $\mathbb{Z}_2^{r-4} \subseteq \mathbb{Z}_2^r$, the fixed point component F of $\mathbb{Z}_2^{r-4}$ containing p satisfies $F \simeq_{h.e.} \mathbb{S}, \mathbb{RP}, \mathbb{CP}$, or $\mathbb{S}/\mathbb{Z}_m$; or
2. M is an integeral cohomology $\mathbb{HP}^{\frac{n}{4}}$ and $r = \frac{n}{4} + 2$.

Example

$\text{Isom}(\mathbb{HP}^{r-2}) \cong \text{Sp}(r-1)/\mathbb{Z}_2$ ($\mathbb{Z}_2 = \{\pm I\}$).

- $\mathbb{Z}_2^{r-2} \cong \mathbb{Z}_2^{r-1}/\mathbb{Z}_2$.
- $\text{Sp}(1) = \{\text{diag}(q, q \ldots, q) : q \in \mathbb{H}, |q| = 1\}$.
- $\text{Sp}(1) \supseteq Q_8$, $Q_8/\{\pm I\} \cong \mathbb{Z}_2^2 = \langle i, j \rangle$.
- $\mathbb{Z}_2^r \cong \mathbb{Z}_2^{r-2} \times \mathbb{Z}_2^2 \subseteq \text{Isom}(\mathbb{HP}^{r-2})$.

$\mathbb{HP}^{r-2} \supseteq \mathbb{HP}^{r-3} \supseteq \ldots \supseteq \mathbb{HP}^2 \supseteq \mathbb{CP}^2 \supseteq \mathbb{RP}^2 \supseteq \mathbb{RP}^1 \supseteq \mathbb{RP}^0$.

Non-example (quarter-maximal $\mathbb{Z}_2$-torus symmetry)

Setup: M – closed, Riemannian, $\sec > 0$.

Theorem (K.K.S.): $\mathbb{Z}_2^r \subseteq \mathrm{Isom}(M^n, g)$, with fixed point, $r \geq \frac{n}{4} + 2$.

1. For every $\mathbb{Z}_2^{r-4} \subseteq \mathbb{Z}_2^r$, the fixed point component F of $\mathbb{Z}_2^{r-4}$ containing p satisfies $F \simeq_{h.e.} \mathbb{S}, \mathbb{RP}, \mathbb{CP}$, or $\mathbb{S}/\mathbb{Z}_m$; or
2. M is an integeral cohomology $\mathbb{HP}^{\frac{n}{4}}$ and $r = \frac{n}{4} + 2$.

Example

$\mathrm{Isom}(\mathbb{CP}^{2k+1}) \cong (\mathrm{Sp}(k+1)/\mathbb{Z}_2) \times \mathbb{Z}_2$

$([z_1, z_2, \ldots, z_{2k+1}, z_{2k+2}] \mapsto [-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{2k+2}, \bar{z}_{2k+1}])$.

- $\mathbb{Z}_2^{\frac{4k+2}{4}+1.5} \cong \mathbb{Z}_2^k \times \mathbb{Z}_2^2 \subseteq \mathrm{Isom}(\mathbb{CP}^{2k+1}/\mathbb{Z}_2)$ (n=4k+2).

Sketch of the proof (half-maximal)

Setup: $\mathbb{Z}_2^r \subseteq \mathrm{Isom}(M^n, g)$ with fixed point p, $r > \frac{n}{2} + 1$.

Goal: M homotopy equivalent to $\mathbb{S}^n$ or $\mathbb{RP}^n$.

- Idea: induction on n.
- $n = 1, 2$, or $3 \Longrightarrow$ done (Fang-Grove).
- Error-correcting codes $\Longrightarrow N := M_p^{\mathbb{Z}_2}$ with $\dim(N) \gtrsim \frac{3}{4} dim(M)$.
- $\mathrm{codim}\,(N) = 1 \Longrightarrow$ reflection groups (Fang-Grove).
- $\mathrm{codim}\,(N) \geq 2 \Longrightarrow$ induction hypothesis applies to N.
- Connectedness lemma $\Longrightarrow$ the conclusion holds for M if it holds for N.

(Connectedness lemma: M – closed, Riemannian, sec > 0, N – totally geodesic submanifold $\Longrightarrow$ the map $\iota^* : H^i(M;\mathbb{Z}) \to H^i(N;\mathbb{Z})$ induced by inclusion is isomorphism for $i \leq 2\dim(N) - \dim(M)$.)

Review of group representations

Setup: V – real vector space, $\dim(V) = n$.

- $\rho : \mathbb{Z}_2^r \to \mathrm{O}(V)$ (orthogonal representation).

- $\rho = \rho_1 \oplus \ldots \oplus \rho_n$, where $\rho_i : \mathbb{Z}_2^r \to \mathrm{O}(\mathbb{R})$ irreducible subrepresentation.

- $V = \oplus_{[\rho_i]} \mathbb{R}^{m_i}$, where $m_i =$ multiplicity of ρ_i.

- $\dim(V) = \sum_{[\rho_i]} m_i$ (Borel formula).

Review of group representations

Setup: V – real vector space, $\dim(V) = n$.

Example

$\rho : \mathbb{Z}_2^2 \to \mathrm{O}(4),\ (z_1, z_2) \mapsto \mathrm{diag}(z_1, z_2, z_1, z_1 z_2)$.
$\rho_1 = \rho_3 : \mathbb{Z}_2^2 \to \mathrm{O}(\mathbb{R})$ ($(z_1, z_2) \mapsto$ multiplication by z_1).
$V = \mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R} \oplus \mathbb{R}$.
$H_1 := \{(1, z) : z \in \mathbb{Z}_2\} \cong \mathbb{Z}_2 \implies Fix(H_1) = \mathbb{R} \oplus 0 \oplus \mathbb{R} \oplus 0$.
$H_2 := \{(z, 1) : z \in \mathbb{Z}_2\} \cong \mathbb{Z}_2 \implies Fix(H_2) = 0 \oplus \mathbb{R} \oplus 0 \oplus 0$.
$H_3 := \{(z, z) : z \in \mathbb{Z}_2\} \cong \mathbb{Z}_2 \implies Fix(H_3) = 0 \oplus 0 \oplus 0 \oplus \mathbb{R}$.

Sketch of the proof (quarter-maximal)

Setup: $\mathbb{Z}_2^r \subseteq \mathrm{Isom}(M^n, g)$ with fixed point p, $n > 12$, $r \geq \frac{n}{4} + 2$,
$\exists\ \mathbb{Z}_2^{r-4} \subseteq \mathbb{Z}_2^r$ such that $F := M_p^{\mathbb{Z}_2^{r-4}} \not\simeq_{h.e.} \mathbb{S}, \mathbb{RP}, \mathbb{CP}$, or lens space.

Goal: M integral cohomology $\mathbb{HP}^{\frac{n}{4}}$.

- $\rho : \mathbb{Z}_2^{r-4} \to \mathrm{O}((T_pF)^\perp)$ (isotropy representation).
  Borel formula: $$\operatorname{codim}(F \subseteq M) = \sum_{\mathbb{Z}_2^{r-5} \subseteq \mathbb{Z}_2^{r-4}} \operatorname{codim}(F \subseteq M_p^{\mathbb{Z}_2^{r-5}}).$$
- $\mathbb{Z}_2^{r-4}$-action effective $\Longrightarrow \#$ (inequivalent subrepresentations) $\geq r - 4$.
- If $\operatorname{codim}(F \subseteq M_p^{\mathbb{Z}_2^{r-5}}) = 1, 2,$ or $3 \Longrightarrow$ contradiction.
  - $\operatorname{codim}(F \subseteq M_p^{\mathbb{Z}_2^{r-5}}) = 1 \Longrightarrow F \simeq_{h.e.} \mathbb{S}$ or $\mathbb{RP}$.
  - $\operatorname{codim}(F \subseteq M_p^{\mathbb{Z}_2^{r-5}}) = 2 \Longrightarrow F \simeq_{h.e.} \mathbb{CP}$ or lens space.
  - $\operatorname{codim}(F \subseteq M_p^{\mathbb{Z}_2^{r-5}}) = 3 \Longrightarrow F \simeq_{h.e.} \mathbb{S}$ or $\mathbb{RP}$.

Sketch of the proof (quarter-maximal)

Setup: $\operatorname{codim}(F \subseteq M_p^{\mathbb{Z}_2^{r-5}}) \geq 4$, $F \not\simeq_{h.e.} \mathbb{S}, \mathbb{RP}, \mathbb{CP}$, or lens space.

Goal: M integral cohomology $\mathbb{HP}^{\frac{n}{4}}$.

- $\operatorname{codim}(F \subseteq M) \geq 4(r-4) \geq n-8 \Longrightarrow \dim(F) \leq 8$.
- If $\dim(F) \leq 7 \Longrightarrow$ contradiction.
- $\dim(F) = 8 \Longrightarrow \begin{cases} \operatorname{codim}(F \subseteq M_p^{\mathbb{Z}_2^{r-5}}) = 4 \\ \#(\text{inequivalent irreducible subrepresentations}) = r-4 \end{cases}$

$$\Longrightarrow \rho = \operatorname{diag}(\pm I_4, \pm I_4, \ldots, \pm I_4).$$

$$\Longrightarrow F \sim \mathbb{HP}^2.$$

- $F^8 \subseteq G^{12} \subseteq H^{16} \subseteq \ldots \subseteq N^{n-4} \subseteq M^n$.
- $\mathbb{HP}^2 \subseteq \mathbb{HP}^3 \subseteq \ldots \subseteq \mathbb{HP}^{\frac{n}{4}-1} \subseteq \mathbb{HP}^{\frac{n}{4}}$.